ISBN: 978-3-95450-179-3 doi:10.18429/JACoW-FEL2017-FRB04



# CANONICAL FORMULATION OF 1D FEL THEORY REVISITED, QUANTIZED AND APPLIED TO ELECTRON EVOLUTION*

Petr M. Anisimov†, Los Alamos National Laboratory, Los Alamos, USA


*Abstract*

An original free-electron laser (FEL) paper relied on quantum analysis of photon generation by relativistic electrons in alternating magnetic field [1]. In most cases, however, the system of pendulum equations for non-canonical variables and the theory of classical electromagnetism proved to be adequate. As x-ray FELs advance to higher energy photons, quantum effects of electron recoil and shot noise has to be considered. This work presents quantization procedure based on the Hamiltonian formulation of an x-ray FEL interaction in 1D case. The procedure relates the conventional variables to canonical coordinates and momenta and does not require the transformation to the Bambini-Renieri frame [2]. The relation of a field operator to a photon annihilation operator reveals the meaning of the quantum FEL parameter, introduced by Bonifacio, as a number of photons emitted by a single electron before the saturation takes place [3]. The quantum description is then applied to study how quantum nature of electrons affects the startup of x-ray FEL and how quantum electrons become indistinguishable from a classical ensemble of electrons due to their interaction with a ponderomotive potential of an x-ray FEL.


## INTRODUCTION

A one dimensional free-electron laser (FEL) theory has played a dominate role in understanding how FELs generate electromagnetic radiation in an undulator with a strength parameter $K = eB_0/k_u m_0 c^2$, which is given in CGS units here, and period $\lambda_u$. This theory allows for an universal scaling that only depends on the FEL parameter $\rho = (1/\gamma)\left(K\Omega_P/4ck_u\right)^{2/3}$ [4] and predicts that in a helical undulator, electrons with energy $\gamma$ in $m_0c^2$ units generate radiation at a wavelength $\lambda = \lambda_u/2\gamma^2\left(1+K^2\right)$. This generation is driven by electron bunching and is governed by the first order equation deduced from Maxwell's equations:

$$\frac{dA}{dz} = \frac{1}{N}\sum_{\alpha=1}^{N} e^{-i\theta_\alpha}, \tag{1}$$

where the field amplitude $A$ is measured in terms of the saturation value $E_s = (m_0c/e)\,\Omega_P\sqrt{\rho\gamma}$, time is replaced by the distance along the undulator $z = ct/L_{g0}$ measured in the units of the gain length $L_{g0} = (2k_u\rho)^{-1}$, and $\theta_\alpha = (k + k_u)\,z_\alpha - \omega t$ is a ponderomotive phase of the $\alpha^{\text{th}}$ electron out of $N$ with respect to the radiation. The electron bunching by the generated radiation is described by the pendulum equations [5, 6] derived most often form the Lorentz force equation:

$$\frac{d\theta_\alpha}{dz} = \eta_\alpha \tag{2a}$$

$$\frac{d\eta_\alpha}{dz} = -2\mathrm{Re}\left(E e^{i\theta_\alpha}\right), \tag{2b}$$

where $\eta_\alpha = (\gamma - \gamma_r)/\rho\gamma_r$ is the relative energy detuning.

Future x-ray FEL designs, that reduce energy of electrons for a given energy of x-ray photons by reducing the undulator period, will require the quantum theory of FEL operation [7]. The equations above are not suitable for a quantum description since they assume that one can specify the exact ponderomotive phase, energy detuning and the field amplitude simultaneously at any point in time. Yet, the principle of stationary action $S$, which is an attribute of the dynamics of a physical system, from which the equations of motion of the system can be derived is better suited for generalizations. Moreover, it is best understood within quantum mechanics, where a system does not follow a single path but its behavior depends on all imaginable paths.

The principle of stationary action is a variational principle $\delta S = 0$ that was best formulated by W. R. Hamiltonian in 1834. It has been used on an occasion to describe electrons in a helical undulator [8] but not the generated radiation, which was described by Maxwell's equations. R. Feynman has demonstrated how this principle can be used in quantum calculations by introducing path integrals [9]. We however will use this principle for an FEL system consisting of relativistic electrons and generated radiation in order to derive a non-relativistic Hamiltonian without the Bambini-Renieri frame [2]. We will then generalize the Hamiltonian principle to quantum mechanics through Poisson brackets for canonical variables. We will finally apply this result to the study of quantum evolution of electrons in an FEL in order to determine if quantum uncertainty of an electron's position can reduce the electron bunching and degrade FEL performance.

## HAMILTONIAN PRINCIPLE

The Hamiltonian principle is W. R. Hamilton formulation of the principle of stationary (least) action. It states that the dynamics of a physical system is determined by a variational problem for a functional based on a single function, the Lagrangian:

$$\delta S\left[\mathbf{q}(t)\right] = \delta\int_{t_1}^{t_2} L\left(\mathbf{q}(t), \dot{\mathbf{q}}(t), t\right)dt = 0\,. \tag{3}$$

One can use it to obtain equations of motion when applied to the action of a mechanical system such as electrons in an FEL but can be also used to derive Maxwell's equations.

* Work supported by LDRD-ER
† petr@lanl.gov

ISBN: 978-3-95450-179-3 doi:10.18429/JACoW-FEL2017-FRB04

First, we write the action function for $N$ relativistic electrons interacting with an electromagnetic field [10]:

$$S = -\sum_{\alpha=1}^{N} \int m_0 c ds_\alpha - \int \frac{A_k j^k}{c^2} d\Omega - \int \frac{F_{ik} F^{ik}}{16\pi c} d\Omega, \quad (4)$$

where $ds_\alpha$ is the proper time interval for the $\alpha^{\text{th}}$ electron; $A_k = (0, -\mathbf{A}_u - \mathbf{A}_r)$ is the four-vector potential in radiation gauge; $j^k$ is the current four-vector, which is the product of the charge density,

$$\rho_e = -e \sum_{\alpha=1}^{N} \delta(\mathbf{r} - \mathbf{r}_\alpha(t)),$$

and the four-velocity vector $u^k = (c, \dot{\mathbf{r}})$; and $d\Omega = c dt d^3\mathbf{r}$ is the proper volume. Finally, $F^{ik} = (\partial \mathbf{A}_r / c\partial t, \nabla \times \mathbf{A}_r)$ is the electromagnetic field tensor in a short-hand notation.

Second, we determine the Lagrangian for the whole system $S = \int_{t_1}^{t_2} (L_\text{e} + L_\text{int} + L_\text{r})\, dt$ after integrating Eq. (4) over interaction volume $V$. The first two terms are standard and describe $N$ relativistic electrons in the presence of an undulator field and generated radiation:

$$L_\text{e} + L_\text{int} = -\sum_{\alpha=1}^{N} \left[ m_0 c^2 \sqrt{1 - \frac{\dot{\mathbf{r}}_\alpha^2}{c^2}} + \frac{e}{c} \left( \mathbf{A}_u^\alpha + \mathbf{A}_r^\alpha \right) \dot{\mathbf{r}}_\alpha \right], \quad (5)$$

where the vector potentials $\mathbf{A}_u^\alpha$ and $\mathbf{A}_r^\alpha$ are evaluated at the position of the $\alpha^{\text{th}}$ electron.

The last term in the Lagrangian describes the generated radiation and is necessary for deriving Maxwell's equations but can be used for deriving the growth equation 1 instead. Let us focus on 1D theory and neglect the dependence of the generated radiation on transverse coordinates $x$ and $y$ in our choice of the vector potential:

$$\mathbf{A}_r(z,t) = -\frac{i}{2k} E(t) \epsilon e^{ikz - i\omega t} + c.c., \quad (6)$$

where $\epsilon$ is a polarization vector and $E(t)$ is a complex, slowly-varying amplitude, which equation of motion we are interested in finding. The corresponding Lagrangian for such a specific radiation in a volume $V$ is

$$L_\text{r} = i \frac{V E^*}{4\pi\omega} \dot{E}, \quad (7)$$

where $dt$ integration by parts on terms containing $\dot{E}^*$ has been carried out; and the term proportional to $\ddot{E}(t)/\omega^2$ has been neglected according to the slowly-varying amplitude approximation.

Finally, the Hamiltonian principle for a 1D FEL can now be written as:

$$\delta \int_{\mathbf{q}(t_1)}^{\mathbf{q}(t_2)} p_E dE + \sum_{\alpha=1}^{N} (\mathbf{p}_\alpha d\mathbf{r}_\alpha - H_\alpha dt) = 0, \quad (8)$$

where the conjugate momenta have been found from $p \overset{\text{def}}{=} \partial L / \partial \dot{q}$ such that

$$p_\alpha = \frac{m \dot{\mathbf{r}}_\alpha}{\sqrt{1 - \dot{\mathbf{r}}_\alpha^2 / c^2}} - \frac{e}{c} \left( \mathbf{A}_u^\alpha - \mathbf{A}_r^\alpha \right)$$

and

$$p_E = i \frac{V}{4\pi\omega} E^*,$$

and $H_\alpha$ is the standard Hamiltonian of an electron in an electromagnetic field obtained as a result of the Legendre transformation of the Lagrangian: $H = \mathbf{p}\dot{\mathbf{q}} - L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)$.

It is common, at this point, to perform the Lorentz transformation from the laboratory frame of reference to the moving frame introduced by Bambini and Renieri in order to obtain a non-relativistic Hamiltonian [2]. In contrast to this approach, we will employ canonical transformations that generate FEL variables and obtain the Bambini-Renieri Hamiltonian for electrons that are near an FEL resonance. This is similar to the approach discussed in Ref. [11] yet it will be carried out in the context of a planar undulator.

## CANONICAL TREATMENT OF PLANAR UNDULATOR

We will assume a near-axis magnetic field of a planar undulator to be $\mathbf{B} = -B_0 \sin(k_u z)\hat{\mathbf{e}}_y$. The Hamiltonian equations of motion immediately imply that the components $p_x = 0$ and $p_y = 0$ are constants of motion. Hence, a single electron Hamiltonian becomes

$$H_\alpha = c\sqrt{p_\alpha^2 + m^2 c^2 \left[1 + \frac{K^2}{2 + K^2} \cos(2k_u z_\alpha)\right]}, \quad (9)$$

where $H_\alpha = \gamma_\alpha(0)\, m_0 c^2$ is the energy of an electron at an undulator entrance, $m^2 = m_0^2 \left(1 + K^2/2\right)$ is a mass of an "undulator" electron that incorporates the transverse degrees of freedom such that $\gamma_{z,\alpha} = \gamma_\alpha(0) / \sqrt{1 + K^2/2}$ with $K = 0.934\, B_0[\text{T}]\, \lambda_u[\text{cm}]$, and $p_\alpha$ is a projection of the canonical momentum, which is no longer a constant of motion, on $z$-axis.

In the limit of $K/\gamma \ll 1$, the projection of the canonical momentum on the axis of a planar undulator has two separate terms:

$$p \approx \bar{p} - \frac{1}{2\bar{p}} \frac{m^2 c^2 K^2}{2 + K^2} \cos(2k_u z), \quad (10)$$

with $\bar{p} = mc\sqrt{\gamma_z^2 - 1}$ as an undulator averaged part of the canonical momentum. The complexity of a planar undulator description comes from the oscillating term so we will remove it by introducing a new canonical momentum with the help of the following generating function

$$\begin{aligned} F_2(z, \bar{p}) &= \int p dz \\ &= z\bar{p} - \frac{1}{4k_u \bar{p}} \frac{m^2 c^2 K^2}{2 + K^2} \sin(2k_u z), \end{aligned} \quad (11)$$

that generates a new canonical coordinate as follows:

$$\bar{z} = \frac{\partial}{\partial \bar{p}} F_2(z, \bar{p}) = z + \frac{1}{4k_u \bar{p}^2} \frac{m^2 c^2 K^2}{2 + K^2} \sin(2k_u z). \quad (12)$$

This leads to a new Hamiltonian $H_\alpha \approx c\sqrt{m^2 c^2 + \bar{p}_\alpha^2}$ where $\bar{p}_\alpha$ is a constant of motion.

ISBN: 978-3-95450-179-3 doi:10.18429/JACoW-FEL2017-FRB04



Generation of electromagnetic radiation perturbs this Hamiltonian such that:

$$H_\alpha = c\sqrt{m^2c^2 + \bar{p}_\alpha^2} + V(z_\alpha, E) + O\left(E^2\right), \tag{13}$$

with an interaction potential

$$V(z_\alpha, E) = \frac{K}{\gamma_\alpha(0)}\mathrm{Im}\left[\frac{eE}{k}\cos(k_u z_\alpha)\, e^{ikz_\alpha - i\omega t}\right] \tag{14}$$

being still dependent on the old coordinate, $z_\alpha$. We need to use Jacobi-Anger expansion

$$\exp(iY\sin\Phi) = \sum_{n=-\infty}^{\infty} J_n(Y)\exp(in\Phi), \tag{15}$$

in order to write the interaction potential in terms of the new coordinate $\bar{z}_\alpha$

$$V(\bar{z}_\alpha, E) = \frac{K}{2\gamma_\alpha(0)}\sum_{n=-\infty}^{\infty}\widehat{JJ}_n \mathrm{Im}\left[\frac{eE}{k}e^{ik_n\bar{z}_\alpha - i\omega t}\right], \tag{16}$$

where $k_n = k + (2n+1)k_u$ and $\widehat{JJ}_n \approx J_n[Y_+] + J_{n+1}[Y_-]$ with $J_n[]$ being Bessel functions of the first kind and $Y_\pm = -((k \pm k_u)/2k_u\gamma_z^2) \times (K^2/4 + 2K^2)$.

We finally say that an FEL interaction is resonant if $k_n\bar{z}_\alpha - \omega t$ does not depend on time, which is fulfilled if $k = (2n+1)\, k_u \dot{\bar{z}}_\alpha \left(c - \dot{\bar{z}}_\alpha\right)^{-1}$. A fundamental mode corresponds to the case of $n = 0$, which results in the interaction potential similar to the one, a relativistic electron experiences in a helical undulator:

$$V_0(\bar{z}_\alpha, E) = \frac{\hat{K}}{2\gamma_\alpha(0)}\mathrm{Im}\left[\frac{eE}{k}e^{i(k+k_u)\bar{z}_\alpha - i\omega t}\right] \tag{17}$$

where $\hat{K} = \widehat{JJ}_0 K$ is a modified undulator parameter.

## INTRODUCING FEL NOTATIONS

It has been pointed out in the previous section that the ponderomotive phase $\theta_\alpha = (k + k_u)\,\bar{z}_\alpha - \omega t$ of the $\alpha^{\text{th}}$ electron with respect to the radiation is what affects the FEL generation. Hence, we will introduce it as a new canonical coordinate with the help of the following generating function, $F_2(\bar{z}_\alpha, p_\theta) = p_\theta[(k + k_u)\,\bar{z}_\alpha - \omega t]$ that generates a new conjugate momentum $\bar{p} = (k + k_u)\, p_\theta$ and a new Hamiltonian:

$$H_\alpha = c\sqrt{m^2c^2 + (k + k_u)^2\, p_{\theta,\alpha}^2} - \omega p_{\theta,\alpha} + V_0(\theta_\alpha, E), \tag{18}$$

which becomes a conserved quantity. The resonance condition, $\dot{\theta}_\alpha = 0$, now implies that $\partial H_\alpha/\partial p_{\theta,\alpha} = 0$ from the corresponding Hamiltonian equation so that the energy of a resonant electron is $\gamma_r m_0 c^2 = mc^2(k + k_u)/\sqrt{k_u(2k + k_u)}$. The later condition corresponds to the velocity of a resonant electron being $\dot{\bar{z}}_r = \omega/(k + k_u)$, which is in accordance with the resonance condition of the previous section.

The resonant condition also implies that the system is near the minimum of the Hamiltonian. Assuming that any departure due to the interaction potential from the resonance is small, we expand the Hamiltonian near this minimum:

$$H_\alpha = H_r + \frac{k_u(k + k_u)^2}{k\gamma_r m_0}\Delta p_{\alpha,\theta}^2 + \frac{\hat{K}}{2\gamma_r}\mathrm{Im}\left[\frac{eE}{k}e^{i\theta_\alpha}\right], \tag{19}$$

and obtain a non-relativistic Hamiltonian without resorting to the transformation into the Bambini-Renieri frame.

We are finally ready to introduce the FEL parameter $\rho$ as a fraction of total electron energy transfered into electromagnetic energy, $\rho\gamma_r m_0 c^2 N = (E_s^2/4\pi)V$, upon saturation. We will also introduce a new independent variable $\tau = 2\rho k_u ct$ that rescales the Hamiltonian as $H = 2\rho k_u c H_{\text{new}}$ so that

$$H_\alpha = \frac{\Delta p_{\alpha,\theta}^2}{2M} + \frac{em_0\hat{K}E_s}{4Mk_u k^2}\mathrm{Im}\left[Ae^{i\theta_\alpha}\right], \tag{20}$$

where an effective mass, $M \approx \rho\gamma_r m_0 c/k$, and a normalized amplitude $A = E/E_s$ have been introduced, and a constant contribution to the Hamiltonian has been omitted. By comparing the first Hamiltonian equation with Eq. (2a), we can identify that $\eta_\alpha = \Delta p_{\alpha,\theta}/M$ such that the second Hamiltonian equation becomes

$$\frac{d\eta_\alpha}{d\tau} = -\frac{em_0\hat{K}E_s}{4M^2k_u k^2}\mathrm{Re}\left[Ae^{i\theta_\alpha}\right]. \tag{21}$$

Comparing this equation with Eq. (2b), one obtains that the FEL parameter is $\rho = (1/\gamma_r)(\hat{K}\Omega_P/8\omega_u)^{2/3}$ in terms of an electron's plasma frequency $\Omega_P^2 = 4\pi e^2 n_e/m_0$ for electron density $n_e = N/V$. The resulting Hamiltonian:

$$H = \sum_{\alpha=1}^{N}\left(\frac{\Delta p_{\alpha,\theta}^2}{2M} + 2M\mathrm{Im}\left[Ae^{i\theta_\alpha}\right]\right), \tag{22}$$

leads to the final equation of the 1D FEL theory Eq. (1) via its derivative with respect to the conjugate moment $p_A = iMNA^*$.

## QUANTIZATION PROCEDURE

In order to quantize the FEL Hamiltonian we note that, since it is related to the original Hamiltonian principle Eq. (8) via canonical transformations, commutation relations are preserved: $\left[\hat{\theta}, \hat{p}_\theta\right] = i\hbar$ and $\left[\hat{A}, \hat{p}_A\right] = i\hbar$. These relationships can now be used to derive Heisenberg equations for operators, which are visually identical to the classical 1D equations Eqs. (1) and (2).

From the first commutator relationship, one can derive the Heisenberg uncertainty for the electron operators that reads in the FEL notations as:

$$\Delta\theta\Delta\eta \geq \frac{\hbar}{2M} = \frac{1}{2\bar{\rho}}, \tag{23}$$

where $\bar{\rho}$ is the quantum FEL parameter introduced in Ref. [3]. In the limit when the quantum FEL parameter is large, the ponderomotive phase and the energy detuning can be exactly specified at any point in time. Thus, the quantum FEL parameter single-handedly determines how quantum electrons are.

ISBN: 978-3-95450-179-3 doi:10.18429/JACoW-FEL2017-FRB04

For radiation operators $\hat{A}$ and $\hat{A}^\dagger$, the commutator is $\left[\hat{A}, \hat{A}^\dagger\right] = (N\bar{\rho})^{-1}$, which is always small due to a large number of participating electrons, $N$. The photon annihilation and creation operators never commute since $\left[\hat{a}, \hat{a}^\dagger\right] = 1$. Comparing two commutators, one can introduce the photon annihilation operator as $\hat{a} = \sqrt{\bar{\rho}N}\hat{A}$ such that a physical meaning of the quantum FEL parameter becomes apparent since $\left\langle \hat{a}^\dagger \hat{a} \right\rangle = \bar{\rho}N\,|A|^2$. It is the number of photons emitted by a single electron before saturation.

## ELECTRON EVOLUTION

A 1D quantum FEL theory can now be used to study the quantum state evolution of electrons in an FEL. In the case of an FEL with 12 GeV electrons that can generate 120 keV photons, the quantum FEL parameter $\bar{\rho} = 10$ for a typical FEL parameter $\rho = 10^{-4}$. In this case the Heisenberg uncertainty principle states that $\Delta\theta_H \geq 0.05$ for $\Delta\eta = 1$, which is the highest energy uncertainty for an FEL to laze. Therefore, an electron is no longer a point particle but a wave packet. It means that a well-localized electron does not stay localized for long if there is no trapping potential since the free space dispersion cases the wave packet to spread [12].

Let us represent the state of a quantum electron at an FEL entrance by a Gaussian of width $\sigma$:

$$\Psi_{\theta_0,p_0}(\theta,0) = \frac{1}{\sqrt{\sqrt{2\pi}\sigma}} e^{-\frac{(\theta-\theta_0)^2}{4\sigma^2} + i\frac{p_0}{\hbar}\theta}, \tag{24}$$

where $\theta_0$ and $p_0$ are initial position and momentum of an electron. The wave packet spreading means that position uncertainty of an electron increases according to

$$\sigma^2(\tau) = \sigma^2 + \frac{\tau^2}{4\bar{\rho}^2\sigma^2}, \tag{25}$$

where the quantum FEL parameter controls the free space dispersion. Since it takes $\tau = 4\pi$ for an FEL to reach saturation, the minimum position uncertainty at saturation is expected to be $\sigma_s = \sqrt{4\pi/\bar{\rho}} \approx 3.5\bar{\rho}^{-1/2}$. It is achieved if an electron started with $\sigma_m = \sigma_s/\sqrt{2}$, which corresponds to $\Delta\eta_0 = 1/\sqrt{8\pi\bar{\rho}}$ initially [13]. This position uncertainty is higher than the one expected from the Heisenberg uncertainty principle, $\Delta\theta_H$.

Assuming that the radiation field amplitude $A$ is a c-number that is equal to the solution of the classical 1D FEL theory equations, the quantum evolution of this wave packet can be described by the Schroedinger equation:

$$i\frac{\partial\Psi}{\partial\tau} = \left\{-\frac{1}{2\bar{\rho}}\frac{\partial^2}{\partial\theta^2} + 2\bar{\rho}\,\mathrm{Im}\left[A(\tau)\,e^{i\theta}\right]\right\}\Psi(\theta,\tau), \tag{26}$$

where the coordinate representation of the Hamiltonian operator has been used.

The radiation field, which determines properties of the interaction potential, grows exponentially $A(\tau) \propto e^{((i+\sqrt{3})/2)\tau}$. Its magnitude determines the strength of the interaction potential while the phase determines its relative position. The linear growth of the phase implies that the interaction potential is moving with a speed $v = -0.5$. Therefore, a stationary electron has a momentum $\Delta p_\theta = -M/2 = -\hbar\bar{\rho}/2$ with respect to the potential, which corresponds to the kinetic energy $\hbar^{-1}\tilde{K} = \bar{\rho}/8$ that places the electron above the potential for as long as $|A| < 0.0625$. This critical magnitude of the radiation field amplitude is reached at $\tau = \tau_s - \pi$ in accordance with the magnitude growth $\exp\left((\sqrt{3}/2)(\tau_s - \pi)\right)$ and $|A_s| = 1$.

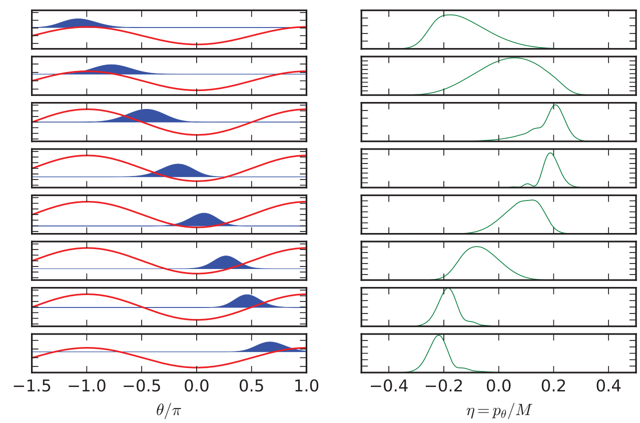


Figure 1: Family of initially stationary wave packets, $p_0 = 0$, with $\sigma_m = 0.354$ and $\theta_0$ in the range from $-\pi$ to $0.75\pi$ (blue) after spending $\tau = 3\pi$ in the interaction potential with $A(0) = 10^{-4}$ and $\bar{\rho} = 50$ (red). The wave packets are presented as $|\Psi|^2$ where a vertical offset corresponds to an average energy of an electron $\left\langle \hat{H} \right\rangle = \int_{-\infty}^{\infty} \Psi^* \hat{H} \Psi d\theta$ that finally places them below the top of the interaction potential. The momentum distributions (green) are no-longer centered at zero, which leads to a net displacement of wave packets and onset of bunching. The wave packet displacements are still masked by the spreading.

Numerical solutions of the Schroedinger equation with $\bar{\rho} = 50$ as in the case of MaRIE x-ray FEL, after interaction time $\tau = 3\pi$ are presented in Figure 1 for a family of initially stationary wave packets, which are uniformly distributed in the range from $-\pi$ to $3\pi/4$. They clearly illustrate our theoretical expectations outlined above. Here, the interaction potential is shifted from its original position by $\psi_A(3\pi) = 3\pi/2$ and is finally strong enough to trap most of the electrons, $\left\langle \hat{H} \right\rangle < 2\bar{\rho}|A(3\pi)|$. At this point, some electrons have already emitted up to 20% of the maximum number of photons yet some of them have absorbed as much thus resulting in $|A(3\pi)|^2 = 0.4\%$ net emission.

The interaction potential is expected to significantly modify evolution of an electron for $\tau > 3\pi$. The electron placed at the most right in Figure 1 will emit the most photons as it continues to increase its negative momentum, which is associated with recoil. Reflection of the positive slope, which moves to the left, will compress that wave packet as well as accelerate it by pushing on it. Thus an electron surfs the ponderomotive wave as it gains momentum.

ISBN: 978-3-95450-179-3 doi:10.18429/JACoW-FEL2017-FRB04

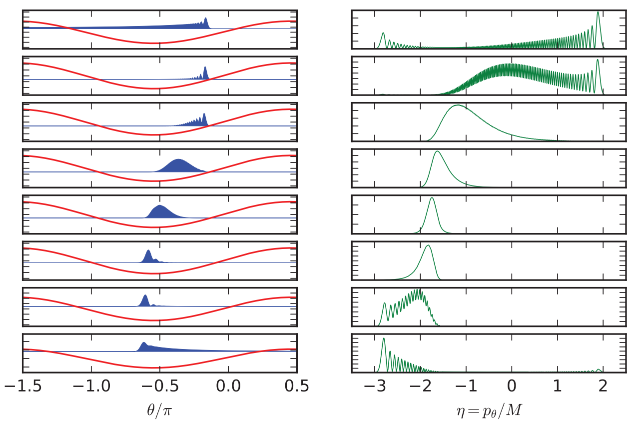


Figure 2: Electron's wave packets (blue) after spending $\tau = 4\pi$ in the interaction potential with $A(0) = 10^{-4}$ and $\bar{\rho} = 50$ (red). At this point, the radiation amplitude reaches its maximum value of $A(4\pi) \approx 1$ that corresponds to the radiation field at saturation. Ensemble averaging of the quantum bunching is $\frac{1}{N}\sum_{\alpha=1}^{N}\left\langle e^{-i\theta_\alpha}\right\rangle = 0.76$, which is equal to the classical bunching.

The wave packets of electrons at saturation are presented in Figure 2. They have been obtained by numerical integration of the Schroedinger equation with $\bar{\rho} = 50$ till the radiation field amplitude has reached its maximum. Here, one can see that the wave packets are no longer Gaussian and may even have long tails that extend beyond a single ponderomotive bucket. The ensemble, however, is bunched within 25% of a single bucket width. Although this bunching is not perfect, $\frac{1}{N}\sum_{\alpha=1}^{N}\left\langle e^{-i\theta}\right\rangle = 0.76$, it is not due to the wave packet spreading but due to different initial positions of electrons as they evolve in the interaction potential.

## CONCLUSION

This manuscript provides canonical formulation of the 1D FEL theory in a planar undulator instead of a helical one. This formulation does not require usage of the Bambini-Renieri frame in order to end up with a non-relativistic Hamiltonian. We have also showed that the growth equation for the generate field amplitude can be derived using Hamiltonian principle instead of Maxwell's equations.

The canonical nature of the theory allows immediate generalization of the Hamiltonian principle to quantum mechanics through Poisson brackets for canonical variables and their relationship to commutators for quantum operators. The quantum analysis of the electron operators identifies the importance of the quantum FEL parameter $\bar{\rho}$ that determines whether a ponderomotive phase and electron's energy can be known exactly at the same time. The quantum analysis of the field operators identifies the physical meaning of the quantum FEL parameter as number of photons emitted by a single electron before saturation.

Finally, the manuscript discusses the quantum state evolution of an electron in an interaction potential created by a classical radiation numerically calculated based on the classical 1D FEL theory. During the first three fourth of the undulator length, the interaction potential is too weak to affect the quantum evolution of an electron state. In the remaining undulator length, the interaction potential dominate the quantum evolution of the electron state, which has been demonstrated by numerically solving the Schroedinger equation for in the case of $\bar{\rho} = 50$ until the radiation amplitude has reached maximum. At this point, electrons are bunched within a 25% of a bucket width by the action of the interaction potential. We thus conclude that reduced bunching cannot be attributed to the wave packet spreading but to different initial positions of electrons with respect to the interaction potential.

## REFERENCES

[1] J. M. J. Madey, "Stimulated emission of bremsstrahlung in a periodic magnetic field," *J. Appl. Phys.*, vol. 42, no. 5, p. 1906, 1971.

[2] A. Bambini and A. Renieri, "Free-electron laser – single-particle classical-model," *Lett. Nuovo Cimento*, vol. 21, no. 11, pp. 399–404, 1978.

[3] R. Bonifacio, N. Piovella, G. R. M. Robb, and A. Schiavi, "Quantum regime of free electron lasers starting from noise," *Phys. Rev. ST Accel. Beams*, vol. 9, no. 9, p. 090701, 2006.

[4] R. Bonifacio, C. Pellegrini, and L. M. Narducci, "Collective instabilities and high-gain regime in a free-electron laser," *Opt. Commun.*, vol. 50, no. 6, pp. 373–378, 1984.

[5] W. B. Colson, "One-body electron dynamics in a free-electron laser," *Phys. Lett. A*, vol. 64, no. 2, pp. 190–192, 1977.

[6] W. H. Louisell, J. F. Lam, D. A. Copeland, and W. B. Colson, "Exact classical electron dynamic approach for a free-electron laser-amplifier," *Phys. Rev. A*, vol. 19, no. 1, pp. 288–300, 1979.

[7] C. Pellegrini, A. Marinelli, and S. Reiche, "The physics of x-ray free-electron lasers," *Rev. Mod. Phys.*, vol. 88, no. 1, p. 55, 2016.

[8] R. Bonifacio, F. Casagrande, and C. Pellegrini, "Hamiltonian model of a free-electron laser," *Opt. Commun.*, vol. 61, no. 1, pp. 55–60, 1987.

[9] R. Feynman and A. Hibbs, *Quantum mechanics and path integrals*. McGraw-Hill, 1965.

[10] L. D. Landau and E. M. Lifshitz, *The classical theory of fields; 3rd ed.*, ser. Course of theoretical physics. Oxford: Pergamon, 1971, Eq. (28.6).

[11] W. Becker and J. K. McIver, "Many-particle quantum theory for a class of free-electron devices," *Physics Reports*, vol. 154, no. 4, pp. 205–245, 1987.

[12] H. P. Freund and T. M. Antonsen, *Principles of Free-Electron Lasers*. Springer US, 1996, page 14, Eq. (1.26).

[13] P. Anisimov, "Quantum nature of electrons in classical x-ray FELs," in *Proc. 37th Int. Free-Electron Laser Conf. (FEL'15)*, 2015, pp. 338–341.