LONGITUDINAL PARTICLE TRACKING CODE
FOR A HIGH INTENSITY PROTON SYNCHROTRON

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Abstract

We have been developing a longitudinal particle tracking code to design and investigate the beam behavior of the J-PARC proton synchrotrons. The code simulates the longitudinal particle motion with a wake voltage and a space charge effect. The code also calculates a longitudinal emittance and a momentum filling factor at an rf bucket under the multi-harmonic wake voltage and the space charge effect. The most different point from the other codes is that a revolution frequency of a synchronous particle is exactly calculated from a bending magnetic field pattern, and it is independent of an acceleration frequency pattern. This feature is useful to check an adiabaticity of the synchrotron. We will describe the specification of the code.

INTRODUCTION

The longitudinal particle tracking code is a powerful tool to investigate the beam behavior in the synchrotron. Although an analytic treatment can be applied for some simple conditions [1], it is very difficult to evaluate the accurate beam behavior under the complicated conditions such as a multi-turn injection, the beam loading effect, the space charge effect, and so on. Many codes has been developed so far and there are some publicly available codes such as ESME [2] and BLonD [3].

We have also been developing a longitudinal particle tracking code for the J-PARC since 1997 [4], and it is aimed at the high intensity proton synchrotron where an acceleration voltage is generated by a Magnetic Alloy (MA) loaded rf cavity [5]. Since the MA cavity has a broadband impedance, we have to pay attention of the beam loading not only for the fundamental acceleration harmonic but also the higher harmonics. The higher harmonics induce the rf bucket distortion, unwanted emittance growth, and they sometimes cause the beam loss.

The J-PARC synchrotrons utilize a multi-harmonic beam loading compensation system by a feedforward method [6, 7]. Our code has been originally developed to estimate how higher harmonics should be compensated. Thereafter, the calculation procedure of the beam emittance and the momentum filling factor under the multi-harmonic beam loading is adapted to check the tolerance of the rf bucket.

We will describe the some basic equations and approaches for simulating the high intensity proton beam. Although the descriptions are only mentioned for the J-PARC, they are also adapted for the other high intensity proton synchrotrons.

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BASIC FEATURES

Our code uses a macro particles of around six thousands per bunch. This number comes from the multi-turn injection at the Rapid Cycling Synchrotron (RCS). The Linac beam is injected into the RCS during 306 turns, and 20 macro particles are injected on each turn. The total number of the macro particles per bunch becomes the multiplication of them.

Difference Equation of Motion

Our code is based on a difference equation of the longitudinal motion [8] as same as the other codes. Our code choose a time from the beginning of the acceleration as the longitudinal coordinate system although a phase of the sinusoidal wave for the fundamental acceleration voltage is sometimes chosen in the other codes. The basic simultaneous equations are:

$$(\Delta E)_{\text{turn}} = eV_{\text{t}}(t) - \Delta E_{\text{s}} \qquad (1)$$

$$(\Delta T_{\text{rev}})_{\text{turn}} = T_{\text{revs}} \left(\frac{1 + \alpha \dfrac{\Delta p}{p_{\text{s}}}}{1 + \dfrac{\Delta \beta}{\beta_{\text{s}}}} - 1 \right) . \qquad (2)$$

The variables in eqs. (1) and (2) are:

$(\Delta E)_{\text{turn}}$	Difference of the energy gain per turn between the arbitrary particle and the synchronous one
$(\Delta T_{\text{rev}})_{\text{turn}}$	Difference of the revolution period per turn between the arbitrary particle and the synchronous one
$V_{\text{t}}(t)$	Total voltage per turn for the arbitrary particle at a time t
ΔE_{s}	Energy gain per turn for the synchronous particle
T_{revs}	Revolution period for the synchronous particle
p_{s}	Momentum of the synchronous particle
Δp	Difference of the momentum between the arbitrary particle and the synchronous one
β_{s}	β of the synchronous particle
$\Delta \beta$	Difference of β between the arbitrary particle and the synchronous one
e	Elementary electric charge
α	Momentum compaction factor

The reason why we choose the time coordinate system is that it is useful to simulate the case that the frequency of the

fundamental acceleration voltage is different from hf_{revs} where h is a harmonic number and f_{revs} is a revolution frequency of the synchronous particle.

Synchronous Particle Tracking

Before tracking the arbitrary particles, we have to know the synchronous particle motion. The revolution period of the arbitrary particle is defined by the momentum compaction factor with the momentum difference between the arbitrary particles and the synchronous particle. Our code adopts a special technique to calculate the synchronous particle.

A popular method to calculate the synchronous particle uses a famous differential equation described by a ramping of a bending magnetic field pattern:

$$\Delta E_{\text{s}} = 2\pi\rho R \frac{dB(t)}{dt} . \tag{3}$$

The variables in eq. (3) are:

ρ	Bending radius of the accelerator ring
R	Average radius of the accelerator ring
$B(t)$	Bending magnetic field at a time t

However, this is an indirect method to track the synchronous particle. Eq. (3) only suggests a instant gradient of the energy gain. If we want to know the energy gain per turn, we must know the revolution period to perform the integration and it is still unknown.

The revolution period of the synchronous particle can be obtained directly by a forward difference method [10] as

$$m_0 c^2 \sqrt{1 + \left(\frac{B(t+T_{\text{revs}})\rho}{m_0 c}\right)^2}$$
$$= \frac{B(t+T_{\text{revs}})c^2\rho}{C}T_{\text{revs}} . \tag{4}$$

The variables in eq. (4) are:

m_0	Rest mass of the particle
c	Speed of the light
C	Circumference of the accelerator ring

Since Eq. (4) is a nonlinear equation, it is solved numerically.

After the revolution period is obtained, we can calculate the energy gain as

$$\Delta E_{\text{s}} = m_0 c^2 \sqrt{1 + \left(\frac{B(t+T_{\text{revs}})\rho}{m_0 c}\right)^2}$$
$$-m_0 c^2 \sqrt{1 + \left(\frac{B(t)\rho}{m_0 c}\right)^2} . \tag{5}$$

Acceleration Voltage Pattern

In our code, the acceleration voltage pattern is designed without any consideration for the high intensity beam issues. The fundamental acceleration voltage V_{rf} and the synchronous phase ϕ_{s} are obtained by the relation as

$$\Delta E_{\text{s}} = V_{\text{rf}} \sin\phi_s . \tag{6}$$

These variables are finally defined independently under the requirement of the beam emittance and the momentum filling factor [9].

The beam emittance ε_{L} is calculated by

$$\varepsilon_{\text{L}} = 2\sqrt{\frac{eV_{\text{rf}}\beta_{\text{s}}^2 E_{\text{s}}}{\pi h^3 \omega_{\text{revs}}^2 \eta_{\text{s}}}} \times$$
$$\int_{\phi_{\text{b1}}}^{\phi_{\text{b2}}} \sqrt{\cos\phi - \cos\phi_{\text{b2}} + (\phi - \phi_{\text{b2})\sin\phi_{\text{s}}}} \, d\phi . \tag{7}$$

The variables in eq. (7) are:

E_{s}	Total energy of the synchronous particle
ω_{revs}	Angular revolution frequency of the synchronous particle
η_{s}	Slippage factor of the synchronous particle
ϕ_{b2}	Maximum phase of the beam emittance
ϕ_{b1}	Minimum phase of the beam emittance

Eq. (7) is integrated numerically by a simple trapezoidal rule in the code. The minimum phase ϕ_{b1} is obtained under the given ϕ_{b2} as

$$\cos\phi_{\text{b1}} - \cos\phi_{\text{b2}} + (\phi_{\text{b1}} - \phi_{\text{b2}})\sin\phi_{\text{s}} = 0 . \tag{8}$$

Furthermore, the momentum filling factor P_{f} with the given ϕ_{b2} is calculated as follows:

$$P_{\text{f}} = \frac{\sqrt{\cos\phi_{\text{s}} - \cos\phi_{\text{b2}} + \sin\phi_{\text{s}}(\phi_{\text{s}} - \phi_{\text{b2}})}}{\sqrt{2\cos\phi_{\text{s}} + \sin\phi_{\text{s}}(2\phi_{\text{s}} - \pi)}} . \tag{9}$$

Note that the filling factor does not depend on the amplitude of the acceleration voltage and it only depends on the injected bunch width. The design value of the filling factor is one of the issues for the longitudinal beam motion. From the experience of the simulation, it is set about 80-90 % for the J-PARC synchrotrons to prevent from the beam loss caused by the beam loading.

In order to make the simulation time shorter, the acceleration voltage pattern, the energy gain and the revolution period of the synchronous particle are prepared by another code previously and the tracking code reads them thereafter.

Voltage Tracking

Another special feature of our code is adapting a voltage wave length tracking. The code tracks not only the revolution period of the particles in eq. (2) but also the wave length of the acceleration voltage. The reason is already mentioned previously. We can simulate using the arbitrary frequency pattern which is different from hf_{revs}.

Typical example is adding a momentum offset at the RCS injection to make a bunching factor larger [10]. In this case, the frequency offset with respect to the synchronous particle is added to the original acceleration frequency pattern.

Beam Current FFT

We have to know the beam harmonics to calculate the wake voltage and the space charge effect. First of all, the arbitrary particles are divided in N bins over one revolution period. The number of bins is 512 for the harmonic number of 2 (RCS) and it is 2048 for the harmonic number of 9 (MR). Since the binning is time consuming procedure, it is better to use a quotient function in stead of the separation by a conditional sentence. After binning, FFT procedure is performed based on the revolution frequency and then we can obtain the beam harmonics.

Wake Voltage

The calculation method of the wake voltage V_b caused by the cavity impedance is quite simple. The beam harmonics I_b obtained by FFT are multiplied by the cavity impedance Z_{cav} as

$$V_b(n\omega_{rev}) = Z_{cav}(n\omega_{rev}) \times I_b(n\omega_{rev}) , \qquad (10)$$

where n is an integer number. It does not matter whether Z_{cav} is an analytic impedance model or a real measurement result. Actually, we evaluate the effect by the parasitic resonance with the real measurement result.

The amplitude and the phase of the wake voltage on each harmonic are calculated on the frequency domain in eq. (10), and then they are reproduced on the time domain by an inverse FFT as

$$V_b(t) = \sum_{n=1}^{N_b} V_{bn}(n\omega_{rev}) \sin(n\omega_{rev}t + \phi_{bn}) , \qquad (11)$$

where ϕ_{bn} is a phase difference of the wake voltage for each harmonic and N_b is the upper limit of the harmonics to be calculated. For example, $N_b = 40$ is chosen for the RCS (~33 MHz) and $N_b = 70$ for the MR (~13 MHz). Since N_b depends on the cavity band width and the bunch width, a heuristic approach is necessary.

Fig. 1 shows the calculation result for 1 MW-eq. beam acceleration in the RCS. The shaded histogram shows the bunch shape, the blue line indicates the fundamental acceleration voltage and the red line indicates the wake voltage. Since the beam intensity is very high, the wake voltage is larger than the fundamental acceleration one. This wake voltage is compensated by the feedforward system.
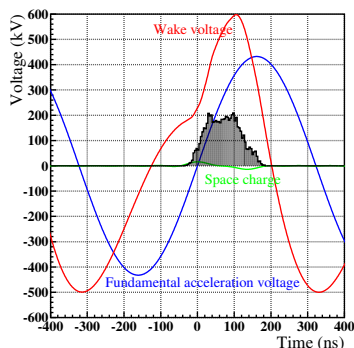

Figure 1: Calculation for the wake voltage (red) and the space charge effect (green) in the RCS.

Space Charge

The space charge effect in the longitudinal direction is calculated by the following formula:

$$V_{sp}(n\omega_{rev}) = -i \cdot nZ_0 \frac{g_0}{2\beta_s\gamma_s^2} \times I_b(n\omega_{rev}) , \qquad (12)$$

where g_0 is a geometrical factor and Z_0 is a characteristic vacuum impedance. The space charge effect in the time domain is obtained by the inverse FFT of eq. (12) as

$$V_{sp}(t) = \sum_{n=1}^{N_{sp}} V_{spn}(n\omega_{rev}) \sin(n\omega_{rev}t + \phi_{spn}) , \qquad (13)$$

where ϕ_{spn} is a phase difference for each harmonic of the space charge voltage and N_{sp} is the upper limit of the harmonics to be calculated.

Defining N_{sp} is an issue because the space charge voltage V_{sp} becomes somewhat noisy waveform if the inverse FFT is performed up to $N/2$. Fig. 2 shows the calculation result of the space charge voltage for the parabolic bunch shape in the RCS where $N_{sp} = 256$ is used. Although the analytic solution should be straight line, the calculation result shows noisy aspect by the binning. From the experience of the simulation [11], we use following relation for N_{sp} as

$$N_{sp} = \frac{2\pi}{\omega_{revs}\sqrt{\sum_{i=1}^{N_P} \Delta t_i^2}} , \qquad (14)$$

where N_P is the number of the macro particles and Δt_i is the time difference of each macro particles with respect to the center of the bunch. Fig. 3 shows the calculation result where $N_{sp} = 10$ (red line), $N_{sp} = 15$ (green line) and $N_{sp} = 30$ (blue line). Eq.(14) suggests $N_{sp} = 14$ is suitable in this case, and the calculation result also suggest $N \sim 15$ is almost enough.
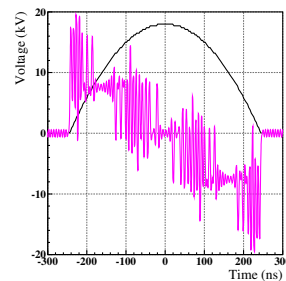

Figure 2: Calculation result of the space charge effect in the case of $N_{sp} = 256$. Black line indicates the parabolic bunch shape.
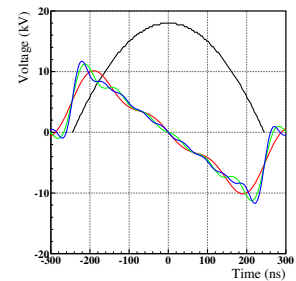
Figure 3: Calculation result of the space charge effect in the case of $N_{sp} = 10$ (red), $N_{sp} = 15$ (green) and $N_{sp} = 30$ (blue).

Beam Emittance and Momentum Filling Factor

The beam emittance and the momentum filling factor mentioned in eq.(7) and (9) are calculated under the condition without the beam loading and the space charge effects.

On the other hand, our code can calculate them including the multi-harmonic beam loading and the space charge effects over whole acceleration period.

The voltage used in this calculation is the total one V_t, and it consists of the fundamental acceleration voltage V_{rf}, the second harmonic voltage V_{rf2} to make the bunch shape flat, the wake voltage V_b and the space charge voltage V_{sp}, that is:

$$
\begin{aligned}
V_t(\phi) =\ & V_{rf} \sin\phi + V_{rf2} \sin\{2(\phi - \phi_s) + \phi_2\} \\
& + \sum_{n=1}^{N_b} V_{bn} \sin\left(\frac{n}{h}\phi + \phi_{bn}\right) \\
& + \sum_{n=1}^{N_{sp}} V_{spn} \sin\left(\frac{n}{h}\phi + \phi_{spn}\right) ,
\end{aligned}
\tag{15}
$$

where ϕ_2 is a phase offset for the second harmonic voltage to adjust the bunch shape.

In that case, the longitudinal beam emittance ε_L is obtained as follows:

$$
\varepsilon_L = 2\sqrt{\frac{e\beta_s^2 E_s}{\pi h^3 \omega_{revs}^2 \eta_s}} \times
$$
$$
\int_{\phi_{b1}}^{\phi_{b2}} \left[\sqrt{\int_{\phi}^{\phi_{b2}} \{V_t(\phi') - V_{rf}\sin\phi_s\}\,d\phi'} \right] d\phi . \tag{16}
$$

Eq. (16) is also integrated numerically by the simple trapezoidal rule. In order to calculate the emittance precisely, we have to find a macro particle on an outermost trajectory under the beam loading and the space charge effect. If the macro particle has a value $(\phi_i, \Delta p_i)$ in the phase space, the maximum phase ϕ_{b2} is obtained from

$$
\frac{e}{\pi h \eta_s E_s \beta_s^2} \int_{\phi_i}^{\phi_{b2}} \{V_t(\phi') - V_{rf}\sin\phi_s\}\,d\phi' - \left(\frac{\Delta p_i}{p_s}\right)^2 = 0
\tag{17}
$$

Once ϕ_{b2} is found, the minimum phase ϕ_{b1} is defined by

$$
\int_{\phi_{b1}}^{\phi_{b2}} \{V_t(\phi') - V_{rf}\sin\phi_s\}\,d\phi' = 0 . \tag{18}
$$

Furthermore, the momentum filling factor P_f is calculated as follows:

$$
P_f(\phi) = \frac{\sqrt{\displaystyle\int_{\phi}^{\phi_{b2}} \{V_t(\phi') - V_{rf}\sin\phi_s\}\,d\phi'}}{\sqrt{\displaystyle\int_{\phi}^{\phi_{bk2}} \{V_t(\phi') - V_{rf}\sin\phi_s\}\,d\phi'}} , \tag{19}
$$

where ϕ_{bk2} is the maximum phase of the rf bucket which is obtained from

$$
V_t(\phi_{bk2}) - V_{rf}\sin\phi_s = 0 . \tag{20}
$$

Fig. 4 shows the total voltage (upper graph) and the rf bucket (lower graph) with (red line) and without (blue line)

the wake voltage at the RCS extraction. Fig. 5 shows the beam emittance and the momentum filling factor with (red line) and without (blue line) the wake voltage over whole acceleration period in the RCS. As can be seen, the higher harmonic beam loading causes the rf bucket distortion and the emittance gradually becomes larger.
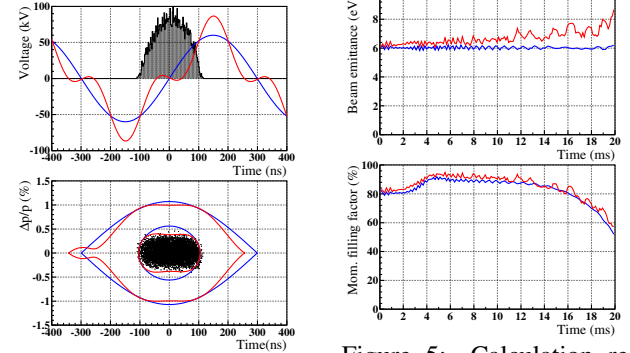

Figure 4: Calculation results for the rf bucket and the trajectory with the beam loading (red) and without the beam loading (blue) in the RCS.
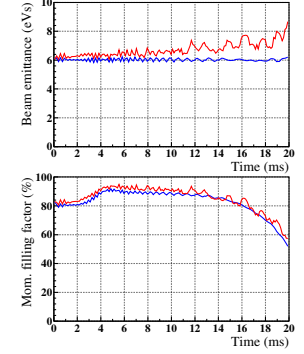

Figure 5: Calculation results for the beam emittance and the momentum filling factor with the beam loading (red) and without the beam loading (blue) in the RCS.

Adiabaticity

It is very important for the longitudinal motion to calculate an adiabaticity ϵ_a. It is defined by

$$
\epsilon_a = \frac{1}{\omega_s^2}\left|\frac{d\omega_s}{dt}\right| , \tag{21}
$$

where ω_s is the angular synchrotron frequency. When $\epsilon_a \ll 1$, typically $\epsilon_a < 0.1$, the change of the accelerating voltage is considered as enough adiabatic in comparison with the synchrotron motion [12]. Since the adiabaticity becomes larger near the extraction in the J-PARC synchrotrons, a frequency correction is necessary to keep the beam orbit center.
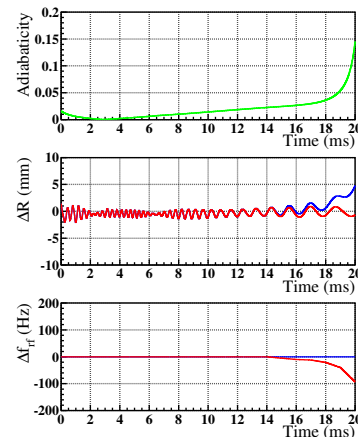

Figure 6: Calculation results for the adiabaticity, the beam orbit and the frequency correction in the RCS.

Fig. 6 shows the calculation results of the adiabaticity (upper graph), the beam orbit (middle graph), and the frequency correction for the acceleration pattern (lower graph) over whole acceleration period of the RCS. Since the adiabaticity reaches around 0.1 near the extraction, the orbit deviates from the center without the frequency correction (blue line). The orbit is corrected by the frequency adjustment (red line).

Phase Modulation

A controlled emittance blow-up is one of the method to solve some high intensity issues. Our code can calculate the controlled emittance blow-up by a phase modulation using Very High Frequency (VHF) cavity [13]. The voltage added by the VHF cavity V_V is expressed as

$$V_V = V_{pm} \sin(h_{pm}\omega_{revs}t + \psi(t) + \psi_{pm}), \qquad (22)$$

where V_{pm} and h_{pm} are an amplitude and a harmonic number of the VHF cavity, and ψ_{pm} is a phase offset value. $\psi(t)$ is a modulation term as follows:

$$\psi_{pm}(t) = \Delta\phi_{pm} \sin \omega_{pm}t, \qquad (23)$$

where $\Delta\phi_{pm}$ is an "amplitude" of the phase modulation and ω_{pm} is an angular modulation frequency. The integer multiple of the synchrotron frequency is usually chosen for the modulation frequency. After all, the total voltage felt by the beam becomes the summation of eqs. (15) and (22). Since the code uses the time domain coordinate, the method of the voltage wave length tracking should be adapted for the VHF voltage. For that purpose, the phase modulation is translated in an instance VHF frequency f_V as

$$f_V(t) = h_{pm}f_{revs} + \frac{\Delta\phi_{pm}}{2\pi}\omega_{pm}\cos\omega_{pm}t . \qquad (24)$$

There is no special procedure to calculate the beam emittance and the momentum filling factor for the phase modulation because the blow-up by the phase modulation is a diffusion process and the trajectory of the particle does not change so much turn by turn.

Fig. 7 shows the beam distribution with (lower graph) and without (upper graph) the phase modulation in the MR injection. The VHF cavity parameters are $V_{pm} = 100$ kV, $h_{pm} = 635$, $\omega_{pm} = 15\omega_s$ and $\Delta\phi_{pm} = \pi$. Fig. 8 shows the beam emittance and the momentum filling factor with (red line) and without (blue line) the phase modulation in the MR injection. As can be seen, the emittance gradually increases from 5 eVs to 10 eVs.

SUMMARY

We have been developing the longitudinal particle tracking code for the high intensity proton synchrotron. It has the special feature for tracking the synchronous particle and the voltage wave length, which is useful to check the adiabaticity. Since the code calculates the beam emittance and the momentum filling factor with the beam loading and the space charge effect, it is worth to evaluate the tolerance in the high intensity beam acceleration.

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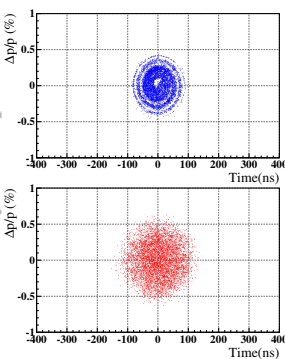[13] M. Yamamoto *et al.*, Proc. of IPAC 2012, WEPPR008 (2012), p. 2952

Figure 7: Calculation result of the longitudinal emittance in the MR with (red) and without the phase modulation (blue).
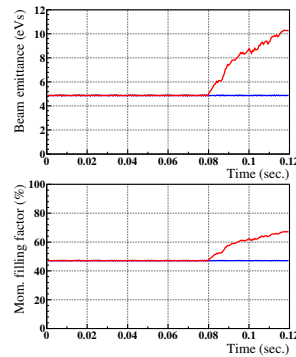

Figure 8: Calculation result of the beam emittance and the momentum filling factor with (red) and without (blue) the phase modulation.