S-Based Space Charge Algorithm for an Electron Gun

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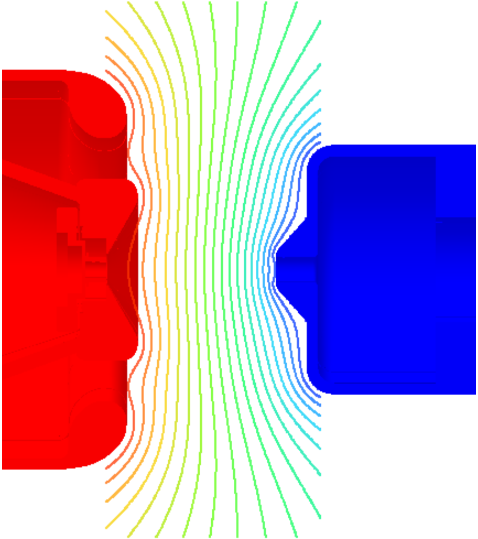
Outline

1. Physical System
2. Continuous Mathematical Model
3. Discrete Mathematical Model

Notation:

$$\dot{\phi} = \partial_t \phi, \qquad \phi' = \partial_z \phi, \qquad \mathbf{x}_\perp = (x, y, 0), \qquad \mathbf{P}_\perp = (P_x, P_y, 0)$$

TRIUMF 300 keV Electron Gun [Ames et al., 2017]



Gap Length	12 cm
Cathode Radius	4 mm
Potential Difference	300 kV
Modulation Frequency	650 MHz
Average Current	10 mA
Maximum Bunch Charge	15.4 pC
Bunch Length	130 ps

TRIUMF 300 keV Electron Gun

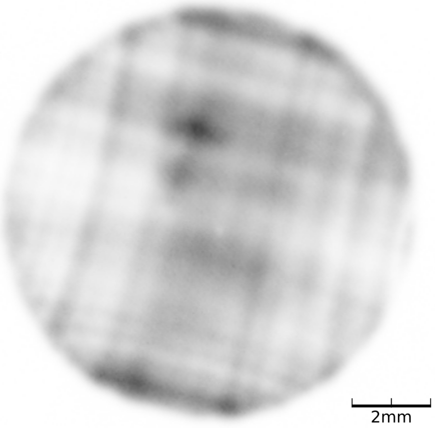


Figure: The view screen image, after the first solenoid

Low Lagrangian

We start from the Low Lagrangian [Low, 1958]:

$$L = \int \mathrm{d}^3\mathbf{x}_0 \mathrm{d}^3\dot{\mathbf{x}}_0 \, \mathcal{L}_p(\mathbf{x}, \dot{\mathbf{x}}; \mathbf{x}_0, \dot{\mathbf{x}}_0, t) + \int \mathrm{d}^3\bar{\mathbf{x}} \, \mathcal{L}_f(\phi, \mathbf{A}; \bar{\mathbf{x}}, t)$$

where:

$$\mathcal{L}_p(\mathbf{x}, \dot{\mathbf{x}}; \mathbf{x}_0, \dot{\mathbf{x}}_0, t) = f(\mathbf{x}_0, \dot{\mathbf{x}}_0) \left(-mc^2 \sqrt{1 - |\dot{\mathbf{x}}|^2/c^2} - q\phi(\mathbf{x}, t) + q\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}, t) \right)$$

$$\mathcal{L}_f(\phi, \mathbf{A}; \bar{\mathbf{x}}, t) = \frac{\epsilon_0}{2} \left(\left| \nabla\phi(\bar{\mathbf{x}}, t) + \dot{\mathbf{A}}(\bar{\mathbf{x}}, t) \right|^2 - c^2 \left| \nabla \times \mathbf{A}(\bar{\mathbf{x}}, t) \right|^2 \right)$$

Relativistic Electrostatic

Rest Frame

$$\Delta\varphi = -\frac{\rho}{\epsilon_0}$$
$$\mathbf{A} = \mathbf{0}$$

Lorentz Transform

$\Longrightarrow$

Lab Frame

$$\phi = \gamma_0 \varphi$$
$$\mathbf{A} = \frac{\beta_0}{c}\gamma_0 \varphi \hat{\mathbf{z}}$$

Relativistic Electrostatic Lagrangian

$$\mathcal{L}_p(\mathbf{x}, \dot{\mathbf{x}}) = -fmc^2\sqrt{1 - |\dot{\mathbf{x}}|^2/c^2} - \frac{fq}{\gamma_0^2}\phi(\mathbf{x}, t)$$

$$\mathcal{L}_f(\phi) = \frac{\epsilon_0}{2}\left(\frac{1}{\gamma_0^2}\left|\nabla_\perp \phi\right|^2 + \left|\phi' + \frac{\beta_0}{c}\dot{\phi}\right|^2\right)$$

Relativistic Electrostatic Potential

The equation of motion, without a source, for ϕ is:

$$\left(\partial_z + \frac{\beta_0}{c}\partial_t\right)^2 \phi + \frac{\beta_0'}{c}\dot{\phi} + (1-\beta_0^2)\nabla_\perp^2\phi = 0$$

$$\beta_0 = 1 \implies \left(\partial_z + \frac{1}{c}\partial_t\right)^2 \phi = 0$$

$$\beta_0 = 0 \implies \phi'' + \nabla_\perp^2\phi = 0$$

Z-Based Lagrangian

We change the independent variable in the Lagrangian with a coordinate transformation.

The new Lagrangian density is:

$$\mathcal{L}_p(\mathbf{x}_\perp, t, \mathbf{x}'_\perp, t'; z) = -fmc\sqrt{(ct')^2 - |\mathbf{x}'_\perp|^2 - 1} - fq\gamma_0^{-2} t' \phi(\mathbf{x}_\perp, t, z)$$

$$\mathcal{L}_f(\phi) = \frac{\epsilon_0}{2}\left(\frac{1}{\gamma_0^2}\left|\nabla_\perp \phi\right|^2 + \left|\phi' + \frac{\beta_0}{c}\dot{\phi}\right|^2\right)$$

Hamiltonian

$$H = \int \mathsf{d}x_0 \mathsf{d}y_0 \mathsf{d}t_0 \mathsf{d}x_0' \mathsf{d}y_0' \mathsf{d}t_0' \, \mathcal{H}_p + \int \mathsf{d}^2 \bar{\mathbf{x}}_\perp \mathsf{d}\bar{t} \, \mathcal{H}_f$$

where:

$$\mathcal{H}_p = -\sqrt{\frac{1}{c^2}\left(E - qf\gamma_0^{-2}\phi(\mathbf{x}_\perp, t, z)\right)^2 - |\mathbf{P}_\perp|^2 - (mfc)^2}$$

$$\mathcal{H}_f = \frac{\pi_\phi^2}{2\epsilon_0} - \frac{\beta_0}{c}\pi_\phi\dot{\phi} - \frac{\epsilon_0}{2\gamma_0^2}\left(\nabla_\perp\phi\right)^2$$

Discreteization Attempt [Webb, 2016]

N Point-like Particles

$$f(\mathbf{x}_0, \dot{\mathbf{x}}_0) = \sum_j w^j \delta^{(3)}(\mathbf{x}_0^j - \mathbf{x}_0)\delta^{(3)}(\dot{\mathbf{x}}_0^j - \dot{\mathbf{x}}_0),$$

where $j = 1, \ldots N$

Fourier Cosine modes in a box $L_x \times L_y \times L_t$

$$\phi(x, y, \Delta t, z) = \sum_{nm\ell} \Phi_{nm\ell}(z) \cos\left(\frac{n\pi x}{L_x}\right) \cos\left(\frac{m\pi y}{L_y}\right) \cos\left(\frac{\ell\pi \Delta t}{L_t}\right)$$

where n, m, ℓ are odd integers.

Discrete Equations of Motion

$$\mathbf{x}_{\perp}^{j\prime} = \frac{\mathbf{P}_{\perp}^{j}}{P_z^j}, \qquad \Delta t^{j\prime} = \frac{E^j - q w^j \gamma_0^{-2} \phi(\mathbf{x}_{\perp}^{j}, t^j, z)}{c^2 P_z^j} + t_0',$$

$$\begin{aligned} \mathbf{P}_{\perp}^{j\prime} &= w^j q \gamma_0^{-2} (t_0' - \Delta t^{j\prime}) \nabla_{\perp} \phi(\mathbf{x}_{\perp}^{j}, t^j, z), \\ \Delta E^{j\prime} &= w^j q \gamma_0^{-2} (t_0' - \Delta t^{j\prime}) \dot{\phi}(\mathbf{x}_{\perp}^{j}, t^j, z) - E_0'. \end{aligned}$$

where the longitudinal momentum is calculated by:

$$P_z^j = -\sqrt{\frac{1}{c^2} \left(E^j - q w^j \gamma_0^{-2} \phi(\mathbf{x}_{\perp}^{j}, t^j, z) \right)^2 - |\mathbf{P}_{\perp}^{j}|^2 - (m w^j c)^2}$$

Discrete Equations of Motion

$$\Phi'_{nm\ell} = \frac{1}{V}\Pi_{nm\ell}$$

$$\begin{aligned}\Pi'_{nm\ell} =& \frac{V}{\gamma_0^2}\left(\left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{\beta_0\gamma_0\ell\pi}{L_t}\right)^2\right)\Phi_{nm\ell} \\ &+ \sum_j \frac{qw^j}{c^2\gamma_0^2}(\Delta t^{j\prime} - t'_0)\cos\left(\frac{n\pi x^j}{L_x}\right)\cos\left(\frac{m\pi y^j}{L_y}\right)\cos\left(\frac{\ell\pi\Delta t^j}{L_t}\right)\end{aligned}$$

where $V = \epsilon_0 L_x L_y L_t/8$

Bad News

- Implementation is unconditionally unstable

Future Work

1. Verify consistency with Lagrangian variational method
   - Try a symplectic integrator
2. Stability analysis of spectral method
3. ???

References

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Thank You