NUMERICAL COMPUTATION OF TRANSPORT MATRICES OF AXISYMMETRIC RF CAVITIES FOR ONLINE BEAM DYNAMICS APPLICATIONS

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Abstract

We describe internally symplectic algorithm for the numerical calculation of the focusing matrices of axisymmetric standing wave TM cavities, which we developed for the online modeling of the beam dynamics in the European XFEL linac.

INTRODUCTION

The RF focusing effect cannot be neglected in many online beam dynamics applications and, especially at relatively low particle energies, has to be taken into account at least in the orbit correction and beam matching programs. Unfortunately, known analytical expressions for the transfer matrix of a cavity are derived under many simplifying assumptions including the assumption that the particle beam is ultra-relativistic. So the actual precision of the analytical models has to be carefully checked for each specific case and, if the accuracy will be found insufficient, a decision has to be made about the further course of action (see [1–5] and references therein for more discussions). It seems that the most universal way out of this situation is to be oriented from the beginning on the numerical computation of the cavity transfer matrices, and in this paper we describe new efficient algorithm for the numerical calculation of the reference particle dynamics and of the 6×6 focusing matrices of the axisymmetric standing wave TM cavities, which we developed for the online modeling of the beam dynamics in the European XFEL linac.

Our integrator utilizes two types of variables: external variables with familiar and well-understood physical sense, and internal variables which are suitable for the application of the symplectic numerical methods. The symplectic numerical methods, which we use, are based on the splitting method (probably, the most frequently used symplectic integrator), which we combine with the techniques of the multiple time scales and of the partial averaging.

LINEAR OSCILLATIONS IN AXISYMMETRIC TM FIELD

As is well known, the linear dynamics in a transverse magnetic (TM) field, which is rotationally invariant with respect to the longitudinal z-direction, is fully determined by the knowledge of the electric field distribution $E_0(t, z)$ along the symmetry axis. The complete set of equations of motion includes nonlinear equations describing dynamics of the reference particle and linear equations of the small oscillations around the ideal reference orbit.

For the accelerating reference particle moving along the z-axis we use, as often, the equations

$$\frac{dt_0}{dz} = \frac{1}{\beta_0 c}, \quad \frac{d\gamma_0}{dz} = \hat{E}_0(t_0, z) \stackrel{\text{def}}{=} \frac{e\, E_0(t_0, z)}{m_0\, c^2}, \tag{1}$$

where γ_0, β_0 and t_0 are the Lorentz factor of the reference particle, its velocity in terms of the speed of light c and its arrival time at a certain position z, respectively.

As concerning transverse dynamics, we consider only the horizontal motion, because due to rotational invariance equations for the both transverse planes are identical.

$$\frac{dx}{dz} = q_x, \tag{2a}$$

$$\frac{dq_x}{dz} = -\frac{1}{\gamma_0 \beta_0^2} \left[\frac{1}{2} \left(\frac{\partial \hat{E}_0}{\partial z} + \frac{\beta_0}{c} \frac{\partial \hat{E}_0}{\partial t} \right) x + \hat{E}_0\, q_x \right]. \tag{2b}$$

Here x measures the horizontal displacement from the symmetry axis and q_x is the horizontal kinetic momentum scaled with the kinetic momentum of the reference particle.

The variables σ and ε which we use for the description of the longitudinal oscillations are

$$\sigma = c\, \beta_0\, (t_0 - t), \qquad \varepsilon = (\gamma - \gamma_0) \,/\, (\beta_0^2\, \gamma_0), \tag{3}$$

and the corresponding equations of motion are as follows

$$\frac{d\sigma}{dz} = \frac{1}{\gamma_0^2} \left(\frac{\hat{E}_0}{\gamma_0 \beta_0^2}\, \sigma + \varepsilon \right), \tag{4a}$$

$$\frac{d\varepsilon}{dz} = -\frac{1}{\gamma_0 \beta_0^2} \left[\frac{1}{\beta_0 c} \frac{\partial \hat{E}_0}{\partial t}\, \sigma + \left(1 + \frac{1}{\gamma_0^2} \right) \hat{E}_0\, \varepsilon \right]. \tag{4b}$$

TRANSFORMATION TO HAMILTONIAN EQUATIONS

In this section we transform equations (2) and (4) to the Hamiltonian form, which enables the possibility to use symplectic methods for their numerical integration. It is clear that such transformations are not unique, and for the equations (2) we use the linear coordinate substitution [6]

$$\begin{bmatrix} x \\ q_x \end{bmatrix} = \frac{1}{\sqrt{\gamma_0 \beta_0}} \begin{bmatrix} 1 & 0 \\ -\frac{\hat{E}_0}{2\gamma_0 \beta_0^2} & 1 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{q}_x \end{bmatrix}, \tag{5}$$

which brings them to the Hill's type form

$$d\tilde{x} \,/\, dz = \tilde{q}_x, \quad d\tilde{q}_x \,/\, dz = -\Omega_X(z)\, \tilde{x} \tag{6}$$

with the Hamiltonian function

$$H_{2X} = H_{2X}^1 + H_{2X}^2, \tag{7a}$$

$$H_{2X}^1 = \tilde{q}_x^2 \,/\, 2, \quad H_{2X}^2 = \Omega_X \tilde{x}^2 \,/\, 2, \tag{7b}$$

where

$$\Omega_X = \left(1 + \frac{2}{\gamma_0^2} \right) \left(\frac{\hat{E}_0}{2\gamma_0 \beta_0^2} \right)^2 - \frac{1}{2\gamma_0^3 \beta_0^3 c} \frac{\partial \hat{E}_0}{\partial t}. \tag{8}$$

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Equations (4) can also be brought to a form similar to the Hill's type form by using the transformation

$$\begin{bmatrix} \sigma \\ \varepsilon \end{bmatrix} = \frac{1}{\sqrt{\gamma_0 \beta_0}} \begin{bmatrix} 1 & 0 \\ -\frac{\hat{E}_0(\gamma_0^2+2)}{2\gamma_0\beta_0^2} & 1 \end{bmatrix} \begin{bmatrix} \tilde{\sigma} \\ \tilde{\varepsilon} \end{bmatrix}, \tag{9}$$

which turn them into the Hamiltonian equations

$$d\tilde{\sigma} \,/\, dz = \tilde{\varepsilon} \,/\, \gamma_0^2, \quad d\tilde{\varepsilon} \,/\, dz = -\Omega_L(z)\, \tilde{\sigma} \tag{10}$$

with

$$\Omega_L = -3\left(1 - \frac{2}{\gamma_0^2}\right)\left(\frac{\hat{E}_0}{2\beta_0^2}\right)^2 - \frac{\gamma_0}{2\beta_0^3 c}\frac{\partial \hat{E}_0}{\partial t}$$
$$-\left(1 + \frac{2}{\gamma_0^2}\right) \cdot \frac{\gamma_0}{2\beta_0^2}\frac{\partial \hat{E}_0}{\partial z}. \tag{11}$$

Unfortunately, though good looking, equations (10) are not well suited for the numerical integration, because Ω_L grows together with the energy increase (stiffness problem) and because Ω_L contains partial derivative of the on-axis field with respect to the spatial variable z, which, in general, can be computed only numerically, i.e. with additional errors.

So we take

$$\begin{bmatrix} \sigma \\ \varepsilon \end{bmatrix} = \frac{1}{\sqrt{\gamma_0 \beta_0}} \begin{bmatrix} \tilde{\sigma} \\ \tilde{\varepsilon} \end{bmatrix} \tag{12}$$

and obtain the system

$$d\tilde{\sigma} \,/\, dz = \partial H_{2L} \,/\, \partial \tilde{\varepsilon}, \quad d\tilde{\varepsilon} \,/\, dz = -\partial H_{2L} \,/\, \partial \tilde{\sigma} \tag{13}$$

with the Hamiltonian function

$$H_{2L} = H_{2L}^1 + H_{2L}^2 + H_{2L}^3, \tag{14a}$$

$$H_{2L}^1 = \left(1 + \frac{2}{\gamma_0^2}\right)\frac{1}{2\gamma_0\beta_0^2}\frac{d\gamma_0}{dz}\,\tilde{\sigma}\tilde{\varepsilon}, \tag{14b}$$

$$H_{2L}^2 = \frac{1}{\gamma_0^2}\frac{\tilde{\varepsilon}^2}{2}, \quad H_{2L}^3 = \frac{1}{\beta_0^3\gamma_0 c}\frac{\partial \hat{E}_0}{\partial t}\frac{\tilde{\sigma}^2}{2}. \tag{14c}$$

STANDING WAVE CAVITY AND NEW REFERENCE PARTICLE VARIABLES

In this paper we are interested in the situation when the scaled on-axis electric field $\hat{E}_0$ has the standing wave form

$$\hat{E}_0(t, z) = \tilde{A}\, a(z)\, \cos(\omega\, t + \varphi_0). \tag{15}$$

We assume that the amplitude function $a(z)$ is defined on the interval $[0,\, l]$ and is normalized in such a way that

$$a_c^2(l) + a_s^2(l) = 1, \tag{16}$$

where

$$a_c(z) + i\, a_s(z) = \int_0^z a(\tau) \exp\left(i\frac{\omega}{c}\tau\right) d\tau. \tag{17}$$

As an example, Fig.1 shows the amplitude function $a(z)$ of the European XFEL TESLA-type 1.3 GHz cavity.

Let us define angle θ by the relations

$$\cos(\theta) = a_c(l), \quad \sin(\theta) = -a_s(l), \tag{18}$$

and let us introduce phase

$$\psi_0(z) = \omega \cdot [t_0(z) - z \,/\, c] + (\varphi_0 - \theta) \tag{19}$$

as a new variable instead of the reference time t_0. In the variables ψ_0 and γ_0 equations (1) take on the form

$$d\psi_0 \,/\, dz = \partial H_R \,/\, \partial \gamma_0, \quad d\gamma_0 \,/\, dz = -\partial H_R \,/\, \partial \psi_0, \tag{20}$$

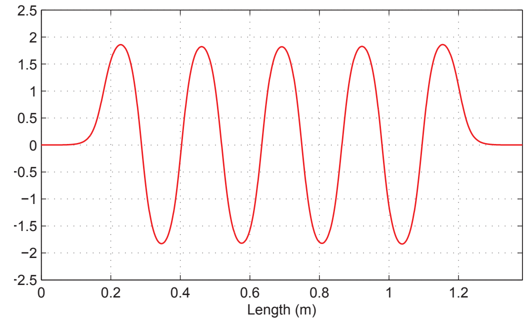

Figure 1: Properly normalized amplitude function $a(z)$ of the European XFEL TESLA-type 1.3 GHz cavity.

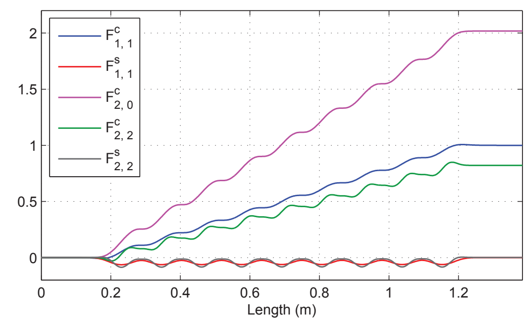

Figure 2: Pre-tabulated functions $F_{n,m}^c$ and $F_{n,m}^s$ calculated for the European XFEL TESLA-type 1.3 GHz cavity.

where

$$H_R = H_R^1 + H_R^2, \tag{21a}$$

$$H_R^1 = \frac{\omega}{c}\left(\sqrt{\gamma_0^2 - 1} - \gamma_0\right), \tag{21b}$$

$$H_R^2 = -\tilde{A}\, a(z)\, \sin\left(\frac{\omega}{c}z + \theta + \psi_0\right). \tag{21c}$$

To clarify the meaning of our normalization of $a(z)$ and the meaning of the new variable ψ_0, let us assume that the reference energy is so high that $\psi_0(z) \approx \psi_0(0)$. Then

$$\gamma_0(l) - \gamma_0(0) \approx \tilde{A}\, \cos(\psi_0(0)). \tag{22}$$

It means that within assumed approximation $\tilde{A}$ is the gain in the γ-factor for the on-crest cavity passage and $\psi_0(0) = 0$ corresponds to the cavity on-crest setup.

SYMPLECTIC INTEGRATION SCHEME

Equations (6), (13) and (20) are Hamiltonian, and we integrate them using symplectic methods constructed on the basis of the well known observation that if an autonomous Hamiltonian H can be decomposed into a sum of n integrable pieces $H = H_1 + \ldots + H_n$ with $\phi_m(h)$ being the corresponding phase flows from z to $z + h$, then the map

$$\phi_1(h/2) \cdots \phi_{n-1}(h/2)\, \phi_n(h)\, \phi_{n-1}(h/2) \cdots \phi_1(h/2) \tag{23}$$

is symmetric second-order symplectic integrator.

Because the composition rule (23) is formally applicable only to the autonomous systems, we first autonomize our Hamiltonians by the standard phase space extension trick. We use one pair of additional conjugated variables (v_z, w_z)

for the autonomization of the Hamiltonian (21) and two pairs, (v_z, w_z) and (v_r, w_r), for the autonomization of the Hamiltonians (7) and (14), because in the latter case the independent variable z enters the Hamiltonians (7) and (14) in two different ways, directly and through the parameters of the reference particle γ_0 and ψ_0, and we would like to reflect and use this difference in our numerical integration scheme.

Numerical Dynamics of the Reference Particle

For the Hamiltonian (21) we use the splitting

$$H_1 = H_R^1, \quad H_2 = H_R^2 + w_z, \tag{24}$$

and obtain the following scheme to advance the variables ψ_0 and γ_0 from z to $z + h$:

$$\psi_0(z + h/2) = \psi_0(z) + \frac{h}{2} \cdot \frac{\omega}{c} \cdot \frac{1 - \beta_0(z)}{\beta_0(z)}, \tag{25a}$$

$$\gamma_0(z + h) = \gamma_0(z)$$
$$+ \tilde{A} \left\{ \cos\left[\psi_0(z + h/2)\right] \left[F_{1,1}^c(z + h) - F_{1,1}^c(z)\right] \right.$$
$$\left. - \sin\left[\psi_0(z + h/2)\right] \left[F_{1,1}^s(z + h) - F_{1,1}^s(z)\right] \right\}, \tag{25b}$$

$$\psi_0(z + h) = \psi_0(z + h/2) + \frac{h}{2} \cdot \frac{\omega}{c} \cdot \frac{1 - \beta_0(z + h)}{\beta_0(z + h)}, \tag{25c}$$

where

$$F_{n,m}^c(z) = \int_0^z a^n(\tau) \cos\left[m\left(\frac{\omega}{c}\tau + \theta\right)\right] d\tau, \tag{26}$$

$$F_{n,m}^s(z) = \int_0^z a^n(\tau) \sin\left[m\left(\frac{\omega}{c}\tau + \theta\right)\right] d\tau. \tag{27}$$

The integrals in (25b) can be approximated by using some quadrature rule of order two or higher (for example, the midpoint rule or the Simpson's rule), but, for the better precision, we prefer to pre-tabulate functions (26) and (27) with highest possible accuracy and then use them as an input for our numerical integrator (see Fig.2).

Numerical Calculation of the Focusing Matrices

For the horizontal and longitudinal oscillations we use the splitting

$$H_1 = H_{2X}^1 + w_r, \quad H_2 = H_{2X}^2 + w_z \tag{28}$$

for the Hamiltonian (7) and the spitting

$$H_1 = H_{2L}^1 + w_r, \quad H_2 = H_{2L}^2, \quad H_3 = H_{2L}^3 + w_z \tag{29}$$

for the Hamiltonian (14), and obtain the propagation rule

$$[\tilde{x}(z + h),\, \tilde{q}_x(z + h)]^\top = X(z, h) \cdot [\tilde{x}(z),\, \tilde{q}_x(z)]^\top, \tag{30a}$$

$$[\tilde{\sigma}(z + h),\, \tilde{\varepsilon}(z + h)]^\top = \Sigma(z, h) \cdot [\tilde{\sigma}(z),\, \tilde{\varepsilon}(z)]^\top, \tag{30b}$$

where X and Σ are the 2×2 matrices with the elements

$$x_{11}(z, h) = x_{22}(z, h) = 1 + h\, u(z, h)\, /\, 2, \tag{31a}$$

$$x_{12}(z, h) = (h\, /\, 2)\; [2 + h\, u(z, h)\, /\, 2], \tag{31b}$$

$$x_{21}(z, h) = u(z, h), \tag{31c}$$

$$\sigma_{11}(z, h) = \kappa(z + h, z)\, d(z, h), \tag{32a}$$

$$\sigma_{22}(z, h) = \kappa(z, z + h)\, d(z, h), \tag{32b}$$

$$\sigma_{12}(z, h) = \kappa(z, z + h/2)\, \kappa(z + h, z + h/2)$$
$$\cdot\, h\, [1 + d(z, h)]\; /\; \left[2\gamma_0^2(z + h/2)\right], \tag{32c}$$

$$\sigma_{21}(z, h) = \kappa(z + h/2, z)\kappa(z + h/2, z + h)\varkappa(z, h), \tag{32d}$$

and abbreviations in (31) and (32) are as follows:

$$u(z, h) = -\frac{\varkappa(z, h)}{2\gamma_0^2(z + h/2)} - \frac{1}{2}\left[1 + \frac{2}{\gamma_0^2(z + h/2)}\right]$$
$$\cdot \left[\frac{B(z, h)}{2}\right]^2 \left\{ \left[F_{2,0}^c(z + h) - F_{2,0}^c(z)\right] \right.$$
$$+ \cos\left[2\psi_0(z + h/2)\right] \left[F_{2,2}^c(z + h) - F_{2,2}^c(z)\right]$$
$$\left. - \sin\left[2\psi_0(z + h/2)\right] \left[F_{2,2}^s(z + h) - F_{2,2}^s(z)\right] \right\}, \tag{33}$$

$$\varkappa(z, h) = \frac{B(z, h)}{\beta_0(z + h/2)}$$
$$\cdot \frac{\omega}{c} \left\{ \cos\left[\psi_0(z + h/2)\right] \left[F_{1,1}^s(z + h) - F_{1,1}^s(z)\right] \right.$$
$$\left. + \sin\left[\psi_0(z + h/2)\right] \left[F_{1,1}^c(z + h) - F_{1,1}^c(z)\right] \right\}, \tag{34}$$

$$B(z, h) = \frac{\tilde{A}}{\beta_0^2(z + h/2)\, \gamma_0(z + h/2)}, \tag{35}$$

$$\kappa(z_1, z_2) = \frac{\beta_0^{3/2}(z_1)\, \gamma_0^{1/2}(z_1)}{\beta_0^{3/2}(z_2)\, \gamma_0^{1/2}(z_2)}, \tag{36}$$

$$d(z, h) = 1 + \frac{h\varkappa(z, h)}{2\gamma_0^2(z + h/2)}. \tag{37}$$

INTEGRATOR SUMMARY

Our integrator utilizes two types of variables: external and internal. External variables are used only as input-output variables and are the variables of the original equations (1), (2) and (4). Internal variables are the variables in which we do actual numerical integration, and transition between external and internal variables is made, whenever it required, according to the formulas (5), (12) and (19).

We haven't found any advantages in using the variable step-size integration and simply calculate the n-step approximation to the focusing matrices transporting beam parameters between the points z_1 and z_2 as the products

$$T_X[z_2]\, X[z_1 + (n - 1)\,\Delta,\, \Delta] \cdots X[z_1,\, \Delta]\, T_X^{-1}[z_1], \tag{38}$$

$$T_L[z_2]\, \Sigma[z_1 + (n - 1)\,\Delta,\, \Delta] \cdots \Sigma[z_1,\, \Delta]\, T_L^{-1}[z_1], \tag{39}$$

where $\Delta = (z_2 - z_1)\, /\, n$, and T_X and T_L are the matrices in the right hand sides of equations (5) and (12), respectively.

Because evaluation of the one-step matrices X and Σ requires knowledge of ψ_0 and γ_0 in three points z, $z + \Delta\, /\, 2$ and $z + \Delta$, for each step of the length $h = \Delta$ in matrix calculations we do two steps of the length $h = \Delta\, /\, 2$ for the reference particle according to the formulas (25).

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