NEW METHOD TO MONITOR THE TRANSVERSE DISTRIBUTION OF CURRENT IN PARTICLE BEAMS

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Abstract

A group of sinusoidally-wound coaxial toroidal coils can be used to determine the transverse distribution of a time-dependent current through their common aperture. The current is expressed in a basis of chapeau (pulse) functions over an array of pixels, and matrix methods are used to calculate the current in each pixel from the voltages induced on the coils. Optimum configurations of pixels are used, for which the condition number of the matrix is bounded by the number of pixels. For example, with 50 pixels the fractional errors in the currents are approximately 50 times the fractional errors in the measured voltages in addition to imperfections in the fabrication and placement of the coils. Numerical tests were made by specifying the currents, calculating the induced voltages, adding Gaussian noise to model measurement errors, and then using the algorithms to calculate the currents. These simulations confirm that the condition number is bounded by the number of pixels.

INTRODUCTION

Others have used a variety of different techniques to monitor the transverse distribution of the beam current in accelerators, including secondary emission monitors, wire scanners, multi-wire chambers, gas curtains or jets, residual gas monitors, scintillator screens, scrapers and measurement targets, synchrotron radiation, and Laser-Compton scattering [1], as well as optical transition radiation [2] and the deflection of a probe beam of electrons [3].

A Rogowski Coil is a non-ferrous current probe formed by bending a uniformly wound helical coil to follow a closed curve having arbitrary shape [4-6]. When a time-dependent current passes through the aperture that is enclosed by the bent helix a voltage is induced on the coil which is independent of the location of the current. However, currents that are located outside of the aperture do not induce a voltage on the coil. Deviations from a uniform winding are carefully avoided because they cause the induced voltage to depend on the location of the current within the aperture, but it will be shown that a group of coils having a specific type of nonuniformity may be used to accurately determine the transverse distribution of the current.

ANALYSIS

Figure 1 is a diagram used for deriving expressions for the open-circuit voltage induced on a non-ferrous toroidal coil that may have a nonuniform winding. The toroid has a mean radius R, and the cross-sectional area of the tube of the toroid is A. Consider the induction in an incremental winding of length Rdθ that is centered at

(R,θ) or equivalently (X_1,Y_1), which is caused by a filament with current $I = I_{10}e^{j\omega t}$ that intersects the X,Y plane at point $P(X_2,Y_2)$. The dashed line L_1 is directed normal to the increment of winding. Dashed line L_2 is parallel to the magnetic field, and dashed line L_3 is parallel to the X-axis.
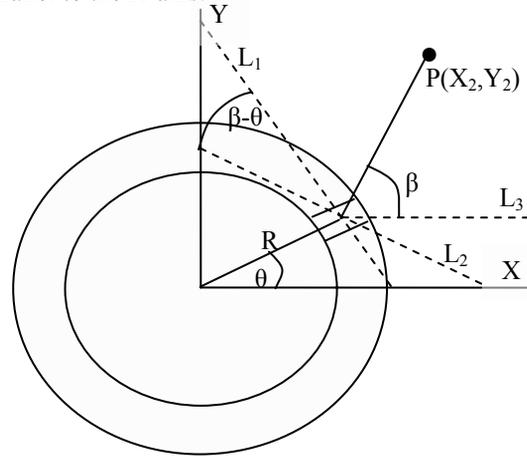

Figure 1: Diagram for analysis.

Let N'(θ) be the number of turns per unit length of the coil, as measured on a circle with radius R. For example, with a toroidal coil having a uniform winding, $N' = N_0' = N_T/2\pi R$, where N_T is the total number of turns. The number of turns in an increment of the winding is $dN = N'(\theta) R d\theta$. Thus, if the height and width of the tube are much less than R, the open-circuit voltage on the entire winding is given by

$$V_{OC} = -\int_0^{2\pi} \frac{j\omega\mu_0 N'(\theta) RAI \cos(\beta-\theta) d\theta}{2\pi\sqrt{(X_2-X_1)^2 + (Y_2-Y_1)^2}} \quad (1)$$

Using trigonometry to obtain an expression for cos(β - θ), Eq. (1) simplifies to give the following:

$$V_{OC} = \frac{j\omega\mu_0 AI}{2\pi} \int_0^{2\pi} \frac{N'(\theta)\left[1 - \frac{X_2}{R}\cos\theta - \frac{Y_2}{R}\sin\theta\right]}{\left[\left(\frac{X_2}{R}-\cos\theta\right)^2 + \left(\frac{Y_2}{R}-\sin\theta\right)^2\right]} d\theta \quad (2)$$

The expressions for N' are chosen from the following set, which constitutes the basis for a Fourier series:

$$N'(\theta) \in \left\langle N_0', \sum_{J=1}^{\infty} N_{JC}' \cos(J\theta), \sum_{J=1}^{\infty} N_{JS}' \sin(J\theta) \right\rangle \quad (3)$$

where the N_{JC}' and N_{JS}' are coefficients as is N_0'. By substituting the set in Eq. (3) into Eq. (2), and evaluating the integral, the open-circuit voltage that is induced on each coil is given by the corresponding term of the

following set:

$$V_{OC} \in j\omega\mu_0 A I_{10} \left\langle \begin{array}{c} N_0{}', \sum_{J=1}^{\infty} \dfrac{N_{JC}{}'}{2}\left(\dfrac{R_2}{R}\right)^J \cos(J\theta_1), \\ \\ \sum_{J=1}^{\infty} \dfrac{N_{JS}{}'}{2}\left(\dfrac{R_2}{R}\right)^J \sin(J\theta_1) \end{array} \right\rangle \quad (4)$$

where $R_2 < R$ so that the current must be located within the aperture.

Expressions for Currents with Known Locations

Consider the case where the locations of M filaments are defined and the currents in each of these filaments are to be determined, or equivalently, the cross-section of the aperture is divided into a group of M pixels which are small enough that the current in each one may be represented by a chapeau (pulse) function. Let the current and the coordinates of the Jth pixel be $I_J = I_{J0}e^{j\omega t}$ and (R_J, θ_J), respectively, for J = 1 to M.

It is convenient to require that M = 2K +1, an odd integer, and require coils with the number of turns per unit length being $N_0{}'$, and both $N_{IC}{}'$ and $N_{IS}{}'$ (cosine and sine terms) having I = 1 to K. That is, there are a total of M coils. Furthermore, define the normalized radial coordinate $S_J = R_J/R$, and the impedances $Z_0 = \omega\mu_0 A N_0{}'$, $Z_{JC} = \omega\mu_0 A N_{JC}{}'/2$, and $Z_{JS} = \omega\mu_0 A N_{JS}{}'/2$, where the reactance of the self inductance of the $N_0{}'$ winding is jZ_0. Thus, Eq. (4) may be generalized to give the following set of equations for the induced voltages:

$$V_{OC} = jZ_0 \sum_{J=1}^{M} I_{J0} \quad (5A)$$

$$V_{OCIC} = jZ_{IC} \sum_{J=1}^{M} S_J{}^I \cos(I\theta_J) I_{J0} \quad (5B)$$

$$V_{OCIS} = jZ_{IS} \sum_{J=1}^{M} S_J{}^I \sin(I\theta_J) I_{J0} \quad (5C)$$

where Eqs. (5B) and (5C) each hold for I = 1 to K. The 3 equations, (5A)-(5C) define a matrix which relates the M measured voltages to the M unknown currents.

Determination of the Ill-Conditioning

Consider the general matrix equation Ax = b, where A is a square matrix and x and b are column vectors. The "direct" problem is defined as solving for the column vector b when A and x are given, and the "inverse" problem is solving for the column vector x when A and b are given. For example, in the present work the direct problem would be calculating the open-circuit voltages when the currents are known and the inverse problem would be calculating the currents when the open-circuit voltages are known.

The errors in solving the inverse problem are frequently much greater than those in solving the forward problem because of what is called "ill-conditioning" of the matrix. It is possible to place an upper bound on the errors in solving the inverse problem as follows:

$$\frac{\|\Delta x\|}{\|x\|} \le \|A\|\|A^{-1}\|\frac{\|\Delta b\|}{\|b\|} \quad (6)$$

where $\|A\|\|A^{-1}\|$, which is the product of the norm of the matrix and the norm of the inverse of the matrix, is defined to be the condition number of matrix A. Several different norms have been used by others, but the Frobenius norm (or Hilbert-Schmidt norm) which is used here is defined as follows [7]:

$$\|A\| = \sqrt{\sum_{I}^{M}\sum_{J}^{M} a_{IJ}{}^2} \quad (7)$$

NUMERICAL TESTS OF ALGORITHMS

Reduction of Ill-Conditioning of the Matrices

Numerical tests were made to determine the condition number for the matrix that is defined by Eqs. (5A)-(5C), using the Frobenius norm with different values of θ_J, S_J, and the impedances.

In the first series of tests the current filaments were placed on a circle with a normalized radius $S_J = 0.5$, and the values of the θ_J were chosen to minimize the condition number, which requires that they are evenly spaced on the circle. Figure 2 shows an example for M = 3. It was determined that the condition number has a minimum value that is equal to M when $Z_{IC} = Z_{IS} = 1.5Z_0/S_J{}^I$ for all values of I so that all of the terms in the matrix have comparable magnitude. For example, with M = 7, the condition number has a value of 7.0 when $Z_{1C} = Z_{1S} = 3.0Z_0$, $Z_{2C} = Z_{2S} = 6.0Z_0$, and $Z_{3C} = Z_{3S} = 12.0Z_0$. For comparison, when $Z_{IC} = Z_{IS} = 0.5Z_0$ for all values of I, the condition number is equal to $2.403 \times 2^{(M+1)/2}$ which is much greater than M when M is large.
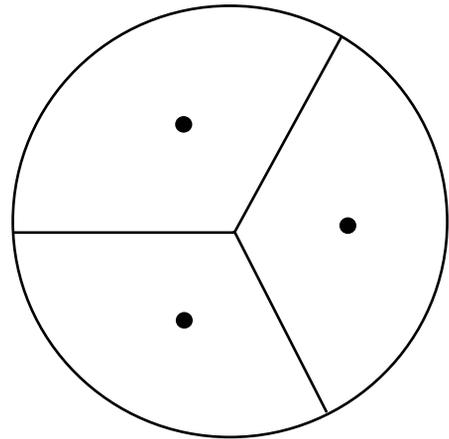

Figure 2: Partitioning of the aperture into 3 sectors.

Further numerical tests were made using pixels that are distributed over the full area of the aperture. Figure 3 shows an example with 9 current filaments that corresponds to the partitioning of a circular aperture into 9 pixels having equal area. In calculations for $Z_{1C} = Z_{1S} = Z_{2C} = Z_{2S} = Z_{3C} = Z_{3S} = Z_{4C} = Z_{4S} = 0.5Z_0$ the condition number equals 155. However, for $Z_{1C} = Z_{1S} = 2.0Z_0$, $Z_{2C} = Z_{2S} = 6.5Z_0$, $Z_{3C} = Z_{3S} = 11.0Z_0$, and $Z_{4C} = Z_{4S} = 23.0Z_0$, so that the matrix elements have comparable magnitude,

the condition number is reduced to a value of 46. The transverse distribution of current could be determined for this array of pixels by using the measurements made with 9 coils.
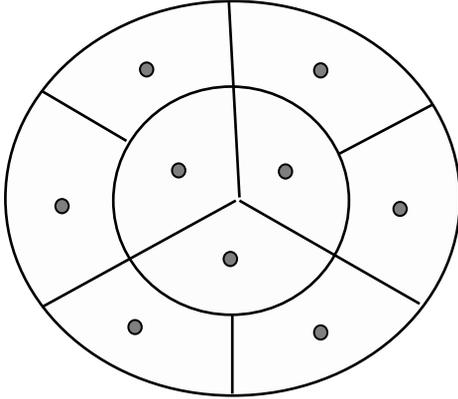

Figure 3: Partitioning of circular aperture into 9 segments.

Numerical Simulations

Numerical simulations were made by specifying the magnitudes, phases, and locations of the currents, calculating the induced voltages, adding Gaussian noise to model measurement errors, and then using the algorithms to calculate the currents and their locations. These results were compared with the input values to find the errors, so the degree of the numerical stability could be determined. Highlights of the results of these tests are as follows:

● For pixels that are evenly spaced on a single circle within the common aperture of a group of toroidal coils, the practical upper limit for the maximum number of pixels is set by the number of coils that can be used for the measurements and not by ill-conditioning of the matrix.

● The maximum number of pixels that may be used is significantly reduced when the pixels are distributed over an area within the aperture.

● It is not surprising that these results are consistent with the results of the numerical tests in which the condition number was determined for the matrices without specifying the magnitudes or phases of the currents.

CONSTRUCTION OF PROTOTYPES

It will be necessary to construct high-precision uniformly-wound and sinusoidally-wound toroidal coils having as many as 100 to 500 turns in order to be able to measure the transverse distribution of current.

Rapid Prototyping methods such as Stereolithography and 3D-Printing could be used to fabricate toroidal forms with grooves in which wires would be placed to form the coils. Typically models made using these techniques have a resolution of 10-100 μm. Prototype toroids having an outer diameter as large as 50 cm could be made with these

methods, and one possible means for fabricating larger toroids would be to use Rapid Prototyping to prepare sections that would be connected together. Rapid Prototyping requires a data file with the (X,Y) coordinates for each Z value corresponding to the height of a given layer of the model.

Determining the Equation for the Coil

The (X,Y) coordinates for each Z value in the data file are determined from the coordinates on the surface of the toroid which are defined in Fig. 4, and the Cartesian coordinates may be determined from the coordinates on the toroid by Eqs. (8)-(10), where R is the mean radius of the toroid and r_0 is the radius of the tube of the toroid.
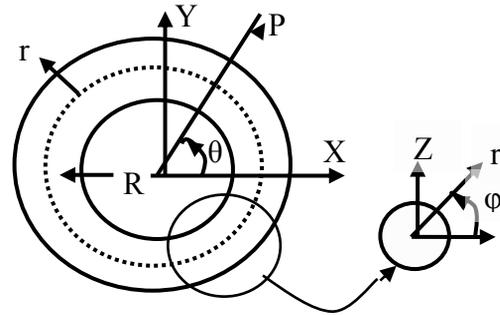

Figure 4: Definition of toroidal coordinates.

$$X = R\cos(\theta) + r_0\cos(\phi)\cos(\theta) \quad (8)$$
$$Y = R\sin(\theta) + r_0\cos(\phi)\sin(\theta) \quad (9)$$
$$Z = r_0\sin(\phi) \quad (10)$$

The equation for the path of the coil in a uniformly-wound coil on the surface of the toroid is given by

$$\phi = 2\pi R N_0'\theta \quad (11)$$

More generally, including non-uniform windings, the path of the coil is defined by

$$\frac{d\phi}{d\theta} = 2\pi R N'(\theta) \quad (12)$$

Thus, by integrating, the equation for the path of the winding for a general set of coils may be defined by extending Eq. (3) as follows:

$$\phi \in \left\langle \begin{array}{c} 2\pi R N_0'\theta, \quad \dfrac{2\pi R N_{JC}'}{J}\sin(J\theta), \\ -\dfrac{2\pi R N_{JS}'}{J}\cos(J\theta) \end{array} \right\rangle \quad (13)$$

Equation (13) may be inverted to derive the following expression which may be used to determine the series of values of θ at which the coil intersects each layer of the model:

$$\theta \in \left\langle \begin{array}{c} \dfrac{\phi}{2\pi R N_0'}, \quad \dfrac{1}{J}Sin^{-1}\left(\dfrac{J\phi}{2\pi R N_{JC}'}\right), \\ \dfrac{1}{J}Cos^{-1}\left(\dfrac{J\phi}{2\pi R N_{JS}'}\right) \end{array} \right\rangle \quad (14)$$

Generating the Data File for Rapid Prototyping

For each value of Z, corresponding to the height of a given layer of the model:

(1) Generate a file of (X,Y) coordinates on a grid with a resolution equal to the incremental step-size for the Rapid Prototyping instrument, such that these points approximate the two circles where the smooth surface of the toroid has the specified Z, requiring that these points have evenly-spaced values of θ.

(2) Define the inner circle, where $\pi/2 < \varphi < 3\pi/2$, and the outer circle, where $0 < \varphi < \pi/2$, or $3\pi/2 < \varphi < 2\pi$.

(3) Determine the first values for φ on each circle:

Inner circle: $\varphi = \pi - \mathrm{Sin}^{-1}(Z/r_0)$

Outer circle: $\varphi = \mathrm{Sin}^{-1}(Z/r_0)$ for $Z > 0$;

$\varphi = 2\pi + \mathrm{Sin}^{-1}(Z/r_0)$ for $Z < 0$.

(4) Use Eq. (14) to determine the corresponding values of θ where the coil crosses the inner and the outer circle.

(5) Add 2π to φ and use Eq. (14) to determine the second pair of θ where the coil crosses the inner and the outer circles.

(6) Continue step 5 to determine the subsequent values of θ until the argument of the inverse trigonometric function exceeds unity. Then use symmetry to determine the other values for θ on the rest of the way around the inner and the outer circles.

(7) Change the data file that was generated in step 1 to include grooves in this layer of the model at the values of θ where the coil crosses the layer.

SUMMARY AND CONCLUSIONS

● It is possible to determine the magnitude and phase of the currents in a number of wires having known locations by measuring the voltages that are induced on an equal number of coaxial toroidal coils that have these currents passing through their common aperture.

● This problem is equivalent to monitoring a continuous distribution of current that passes through a specified area which is divided into a number of pixels. These pixels must be small enough that the current is slowly varying within each of them. The well-known smoothing property of the integral operator makes a partial correction for the effects of the linear variation of the current within each pixel.

● It is convenient to use matrix methods to determine the magnitudes and phases of the currents when the number of wires is an odd integer $M = 2K + 1$, and the coils have the number of turns per unit length constant, and proportional to the $\sin(\theta)$, $\cos(\theta)$, $\sin(2\theta)$, $\cos(2\theta)$, … , $\sin(K\theta)$, and $\cos(K\theta)$, respectively, where θ is the azimuthal coordinate.

● When the pixels are evenly spaced on a single circle within the common aperture of a group of toroidal coils, the practical upper limit for the maximum number of pixels is set by the number of coils that can be used for the measurements and not by ill-conditioning of the matrix. However, the maximum number of pixels that may be used is significantly reduced when the pixels are distributed over an area within the aperture. However, it is not necessary to have pixels in areas of the aperture where it is known that the current is negligible.

● Rapid Prototyping methods have adequate precision for fabricating the group of coils, and the equations that are needed to generate the data files which are required for Rapid Prototyping have already been derived.

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