EXPRESSING PROPERTIES OF BPM MEASUREMENT SYSTEM IN TERMS OF ERROR EMITTANCE AND ERROR TWISS PARAMETERS

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Abstract

We show that properties of the beam position monitor (BPM) system designed for the measurement of transverse and energy beam jitters can be described in terms of the usual accelerator physics concepts of emittance, energy spread, dispersions and betatron functions. Besides that, using the Courant-Snyder quadratic form as error estimator we introduce the scalar objective function which can be used as design criteria of a BPM measurement system with needed properties.

INTRODUCTION

The determination of variations in the beam position and in the beam energy using BPM readings is one of the standard problems of accelerator physics. If the optical model of the beam line and BPM resolutions are known, the typical choice is to let jitter parameters be a solution of the weighted linear least squares problem. For transversely uncoupled motion this least squares problem can be solved analytically, but the direct usage of the obtained solution as a tool for designing a BPM measurement system is not straightforward. A better understanding of the nature of the problem is needed. In this article, following the papers [1, 2, 3], we show that properties of the BPM measurement system can be described in terms of the usual accelerator physics concepts of emittance, energy spread, dispersions and betatron functions. Due to space limitation, we consider only the case of transversely uncoupled nondispersive beam motion and inclusion of the energy degree of freedom and multiple examples can be found in the papers cited above.

STANDARD LEAST SQUARES SOLUTION

We assume that the transverse particle motion is uncoupled in linear approximation and use the variables $\vec{z} = (x,\, p_x)^\top$ for the description of the horizontal beam oscillations. As transverse jitter parameters in the point with the longitudinal position $s = r$ (reconstruction point) we mean the difference $\delta\vec{z}(r) = \vec{z}(r) - \vec{z}_g(r)$ between parameters of the instantaneous orbit and parameters of some predetermined "golden trajectory" $\vec{z}_g = (\bar{x},\, \bar{p}_x)^\top$.

Let us assume that we have n BPMs in our beam line placed at positions $s_1, \ldots, s_n$ and they deliver readings $\vec{b}_c = (b_1^c, \ldots, b_n^c)^\top$ for the current trajectory with previously recorded observations for the golden orbit being $\vec{b}_g = (b_1^g, \ldots, b_n^g)^\top$. Suppose that the difference between these readings $\delta\vec{b}_\varsigma = \vec{b}_c - \vec{b}_g$ can be represented in the form

$$\delta\vec{b}_\varsigma = (x(s_1) - \bar{x}(s_1), \ldots, x(s_n) - \bar{x}(s_n))^\top + \vec{\varsigma}, \qquad (1)$$

where the random vector $\vec{\varsigma} = (\varsigma_1, \ldots, \varsigma_n)^\top$ has zero mean and positive definite covariance matrix

$$V_\varsigma \;=\; \mathrm{diag}\left(\, \sigma_1^2,\, \sigma_2^2,\, \ldots,\, \sigma_n^2 \,\right). \qquad (2)$$

As usual, we find an estimate $\delta\vec{z}_\varsigma(r)$ for the difference orbit parameters $\delta\vec{z}(r)$ in the presence of BPM reading errors $\vec{\varsigma}$ by fitting the difference in the BPM data $\delta\vec{b}_\varsigma$ to the known optical model of the beam line, i.e. by solving weighted linear least squares problem. If the phase advance between at least two BPMs is not a multiple of $180°$, then the result of this fit is unique and is given by the formula

$$\delta\vec{z}_\varsigma(r) = \left(M^\top(r) V_\varsigma^{-1} M(r)\right)^{-1} M^\top(r)\, V_\varsigma^{-1} \cdot \delta\,\vec{b}_\varsigma. \qquad (3)$$

The calculation of the covariance matrix of the reconstruction errors is also standard and gives the following result

$$V_z(r) \;\stackrel{\mathrm{def}}{=}\; \mathcal{V}\left(\, \delta\vec{z}_\varsigma(r)\,\right) \;=\; \left(M^\top(r)\, V_\varsigma^{-1}\, M(r)\right)^{-1}. \qquad (4)$$

Here

$$M \;=\; \begin{pmatrix} a_{11}(r,\, s_1) & a_{12}(r,\, s_1) \\ \vdots & \vdots \\ a_{11}(r,\, s_n) & a_{12}(r,\, s_n) \end{pmatrix}, \qquad (5)$$

and $a_{11}(r_1,\, r_2)$, $a_{12}(r_1,\, r_2)$ are the elements of a two by two symplectic matrix $A(r_1,\, r_2)$ which transfers particle coordinates from the point with the longitudinal position $s = r_1$ to the point with the position $s = r_2$.

For the considered one-dimensional case the matrix inversion in the right hand side of the formula (4) can be done analytically and the elements of the error covariance matrix $V_z(r)$ can be obtained in the explicit form as follows

$$(V_z(r))_{1,1} = \frac{1}{\Delta} \sum_{m=1}^{n} \left(\frac{a_{12}(r, s_m)}{\sigma_m}\right)^2, \qquad (6)$$

$$(V_z(r))_{1,2} = (V_z(r))_{2,1} =$$

$$-\frac{1}{\Delta} \sum_{m=1}^{n} \left(\frac{a_{11}(r, s_m)}{\sigma_m}\right) \left(\frac{a_{12}(r, s_m)}{\sigma_m}\right), \qquad (7)$$

$$(V_z(r))_{2,2} = \frac{1}{\Delta} \sum_{m=1}^{n} \left(\frac{a_{11}(r, s_m)}{\sigma_m}\right)^2, \qquad (8)$$

where

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$$\Delta = \frac{1}{2} \sum_{k,m=1}^{n} \left(\frac{a_{11}(r, s_k)a_{12}(r, s_m) - a_{11}(r, s_m)a_{12}(r, s_k)}{\sigma_k \sigma_m} \right)^2 . \quad (9)$$

In theory, the formulas (6)-(9) contain all information which one has to know in order to be able to design a BPM measurement system with needed properties or in order to be able to compare expected performance of two different measurement systems. In practice, unfortunately, the simple knowledge of formulas (6)-(9) is quite far from being sufficient for these purposes. Let us assume, for example, that we want to compare resolutions of two BPM systems which are supposed to be used for trajectory jitter determination and are installed in two different beam lines. For this purpose we need, at least, to have a criteria how to compare two covariance matrices and to know how to chose the reconstruction points (own for each measurement system) in which such comparison has to be done. Does all that looks to be fairly straightforward?

ERROR TWISS PARAMETERS AND COURANT-SNYDER QUADRATIC FORM AS ERROR ESTIMATOR

An important step in solving the problems marked at the end of the previous section was made in [1, 2], where dynamics was introduced into this problem which in the beginning seemed to be static. When one changes the position of the reconstruction point, the estimate of the jitter parameters propagates along the beam line exactly as a particle trajectory and it becomes possible (for every fixed jitter values) to consider a virtual beam consisting of virtual particles obtained as a result of application of least squares reconstruction procedure to “all possible values” of BPM reading errors. The dynamics of the centroid of this beam coincides with the dynamics of the true difference orbit and the covariance matrix of the jitter reconstruction errors can be treated as the matrix of the second central moments of this virtual beam distribution and satisfies the usual transport equation

$$V_z(r_2) \;=\; A(r_1,\, r_2)\, V_z(r_1)\, A^{\top}(r_1,\, r_2). \quad (10)$$

Consequently, for the description of the propagation of the reconstruction errors along the beam line, one can use the accelerator physics notations and represent the error covariance matrix in the familiar form

$$V_z(r) = \epsilon_\varsigma \begin{pmatrix} \beta_\varsigma(r) & -\alpha_\varsigma(r) \\ -\alpha_\varsigma(r) & \gamma_\varsigma(r) \end{pmatrix}, \quad (11)$$

where $\beta_\varsigma(r)$ and $\alpha_\varsigma(r)$ are the error Twiss parameters and

$$\epsilon_\varsigma \;=\; \sqrt{\det V_z(r)} \;=\; 1/\sqrt{\Delta} \quad (12)$$

is the invariant error emittance.

Note that the error Twiss parameters can also be found as solution of the minimization problem

$$\min_{\beta(r),\,\alpha(r)} \; \sum_{m=1}^{n} \frac{\beta(s_m)}{\sigma_m^2}. \quad (13)$$

Under the assumption that the phase advance between at least two BPMs is not a multiple of 180°, the solution of the problem (13) is unique, the minimum is reached at the error Twiss parameters and is equal to $2/\epsilon_\varsigma$.

Parametrization (11) is an essential step ahead in understanding of the structure of the matrix $V_z(r)$ in comparison with the formulas (6)-(9). It introduces such important characteristic of BPM measurement system as error emittance and shows that balance between coordinate and momentum reconstruction errors in the point of interest is defined by the values of error Twiss parameters at this location. Nevertheless, it still does not give a single property to compare two different BPM systems. Fortunately, the beam dynamical point of view on the BPM measurement system naturally suggests us that in order to obtain the needed criteria we may simply use the Courant-Snyder quadratic form as an error estimator.

Let $\beta_0,\, \alpha_0,\, \gamma_0$ be the design Twiss parameters and

$$I_x(r,\, \vec{z}\,) = \gamma_0(r)\, x^2 + 2\alpha_0(r)\, xp_x + \beta_0(r)\, p_x^2 \quad (14)$$

the corresponding Courant-Snyder quadratic form. Using this quadratic form we introduce the random variable

$$I_x^\varsigma \;=\; I_x(r,\, \delta\vec{z}_\varsigma(r) - \delta\vec{z}(r)). \quad (15)$$

The mean value of this random variable is equal

$$\langle\, I_x^\varsigma \,\rangle \;=\; 2\, \epsilon_\varsigma\, m_p(\beta_\varsigma,\, \beta_0), \quad (16)$$

where

$$m_p(\beta_\varsigma,\, \beta_0) = (\beta_\varsigma\gamma_0 - 2\alpha_\varsigma\alpha_0 + \gamma_\varsigma\beta_0)\,/\,2. \quad (17)$$

The right hand side in (16) does not depend on the position of the reconstruction point (is a number), characterizes the resolution of the BPM system not in some absolute units but in the relative units of beam sigmas and, therefore, allows to compare properties of two completely different BPM systems installed in two different beam lines and also can be used as scalar valued (not matrix valued) design criteria.

TWO BPM CASE

Let us consider two BPMs separated in the beam line by a transfer matrix $A(s_1, s_2)$ with $a_{12} \neq 0$ and assume that these BPMs deliver uncorrelated readings with rms resolutions σ_1 and σ_2 respectively. Often, when one works on optimization of the optics of two BPM system, one speaks about the desire to have the large beta functions at the BPM locations and the phase advance being equal or enough close to 90° . And that is completely right, if one will interpret this desire as a way to increase absolute value of the a_{12} coefficient, because the error emittance of the two BPM measurement system is inversely proportional to it

$$\epsilon_\varsigma \;=\; (\sigma_1\,\sigma_2)\,/\,|a_{12}|\,. \tag{18}$$

Nevertheless, because due to formula (16) the figure of merit for the quality of BPM system is not the error emittance alone, but the product of the error emittance and the mismatch between the error and the design Twiss parameters (large mismatch can spoil the properties of the measurement system even for the case when the error emittance is small), one has to take additional care and compare design betatron functions with the error betatron functions which are given below

$$\beta_\varsigma(s_1) = \frac{\sigma_1}{\sigma_2}\,|a_{12}|\,,\quad \alpha_\varsigma(s_1) = \frac{\sigma_1}{\sigma_2}\mathrm{sign}\,(a_{12})\,a_{11}, \tag{19}$$

$$\beta_\varsigma(s_2) = \frac{\sigma_2}{\sigma_1}\,|a_{12}|\,,\quad \alpha_\varsigma(s_2) = -\frac{\sigma_2}{\sigma_1}\mathrm{sign}\,(a_{12})\,a_{22}. \tag{20}$$

Let us note that though the error Twiss parameters depend on the ratio of BPM resolutions, the error phase advance (phase advance defined by β_ς) is independent from this ratio and is always equal to an odd multiple of 90°.

PERIODIC MEASUREMENT SYSTEM

In this section we consider a measurement system constructed from n identical cells assuming that we have one BPM per cell (identically positioned in all cells) and that the cell transfer matrix allows periodic beam transport with phase advance μ_p being not a multiple of 180°. Additionally, we assume that all BPMs have the same rms resolution σ_{bpm}. In this situation the formula for the error emittance is rather simple and is given by the expression

$$\epsilon_\varsigma \;=\; \frac{2\sigma_{bpm}^2}{n\beta_p(s_1)}\cdot m_p(\beta_\varsigma,\,\beta_p)\,, \tag{21}$$

where $\beta_p(s_1)$ is the value of the periodic betatron function at the BPM locations and

$$m_p(\beta_\varsigma,\,\beta_p) \;=\; \left(1\;-\;\left(\frac{1}{n}\cdot\frac{\sin(n\mu_p)}{\sin(\mu_p)}\right)^2\right)^{-\frac{1}{2}} \tag{22}$$

is the mismatch between the error and the periodic betatron functions (even so we do not assume, in general, periodic betatron functions being the design betatron functions matched to our beam line).

There is a rather widespread opinion that a periodic measurement system reaches an optimal performance when its design Twiss parameters are cell periodic and the cell phase advance is a multiple of 180° divided by n. Is that really so? To answer this question let us take the cell periodic Twiss parameters as design Twiss parameters and write

$$\langle I_x\rangle \;=\; \frac{4\sigma_{bpm}^2}{n\beta_p(s_1,\,\mu_p)}\cdot m_p^2(\beta_\varsigma,\,\beta_p)\,. \tag{23}$$

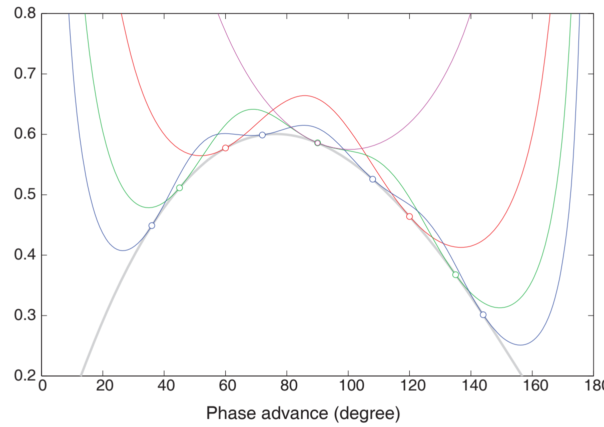

Figure 1: Functions $\Psi_n\,(\mu_p)$ shown for $n = 2,3,4,5$ (magenta, red, green and blue curves respectively). The gray curve shows function $\Psi_\infty\,(\mu_p)$.

Looking at the formula (22) one sees that the choice of μ_p such that $\sin(n\mu_p) = 0$ makes the error and the periodic Twiss parameters equal and brings the second multiplier in the right hand side of the formula (23) to the minimal possible value. But, in general, it does not guarantee that the product of the two multipliers in (23) is also minimized. So the answer is not or, more exactly, not necessary.

To be more specific, let us consider a thin lens FODO cell of the length L as a basic unit of our periodic system and let us also assume that the BPM is placed in the "center" of the focusing lens. In this situation

$$\langle I_x\rangle \;=\; 2\epsilon_\varsigma m_p\,(\beta_\varsigma,\,\beta_p) \;=\; \frac{4\sigma_{bpm}^2}{nL}\cdot\Psi_n\,(\mu_p)\;, \tag{24}$$

where

$$\Psi_n\,(\mu_p) \;=\; \Psi_\infty\,(\mu_p)\cdot m_p^2\,(\beta_\varsigma,\,\beta_p)\,, \tag{25}$$

$$\Psi_\infty\,(\mu_p) \;=\; \sin(\mu_p)\,/\,(1+\sin(\mu_p/2)). \tag{26}$$

The functions Ψ_n for $n = 2,3,4,5$ are plotted in figure 1 together with their values in the points

$$\mu_p \;=\; k\cdot(180^\circ\,/\,n),\quad k = 1,\ldots,n-1, \tag{27}$$

shown as small circles at the corresponding curves. One sees that for all n the optimal performance of our measurement system (minimum of Ψ_n) is reached for the phase advance which is different from the multiples of $180^\circ\,/\,n$.

REFERENCES

[1] V.Balandin, W.Decking and N.Golubeva, "Errors in Reconstruction of Difference Orbit Parameters due to Finite BPM-Resolutions", TESLA-FEL 2009-07, DESY, July 2009.

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