LONGITUDINAL AND TRANSVERSE SELF-FORCES IN RELATIVISTIC LINE-CHARGE BEAM *

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Abstract

An attempt to compute self-fields of a beam with negligible transverse dimensions (a line-charge beam) in general leads to logarithmically diverging results. At the same time, the well known coherent synchrotron radiation wake [1] does not involve the transverse size of the beam and is applicable to beams with infinitely small transverse dimensions. The result of Ref. [1], however, is limited to steady state motion in a uniform magnetic field.

In this work, we formulate equations of motion in both longitudinal and transverse directions in such a way that they allow transition to the limit of negligible transverse dimensions of the bunch, and can be applied to a line-charge model of the beam. The developed formalism can be used, in particular, to simplify calculations of the CSR effects in bunch compressors.

INTRODUCTION

Beam dynamics of short bunches with high peak current is an important element of modern light sources, including free electron lasers. The effect of coherent synchrotron radiation (CSR) of such bunches, and the fields that it generates inside the beam, has been a subject of intensive theoretical and experimental studies for the last two decades. The existing computer codes for simulation of the CSR effects are now widely used in the design of systems with high brightness, high peak current beams [2, 3].

There has also been a noticeable progress in theoretical studies of the CSR effects [4, 5]. The success of the theoretical analysis [5, 6] clarified such important issues in the CSR theory as a so called *cancellation effect*, raised in one of the first publications on the subject [7]. The cancellation effect refers to the compensation of terms in the transverse equation of motion logarithmically diverging with diminishing transverse size of the beam by similar terms in the electrostatic potential of the self field.

In regard to this cancellation effect, a question arises if one can formulate a self consistent system of equations of motion for the beam, and equations for the electromagnetic field, which are applicable to the beam with *zero transverse dimensions*? In other words, if $\sigma_\perp$ is the transverse size of the beam, is there a finite limit for observable quantities characterizing the beam when $\sigma_\perp \to 0$? This is not a trivial question. The results of Refs. [1, 8] do not involve the transverse size and suggest that such limit might exist. These results, however, are derived for steady state motion in a uniform magnetic field only. With some additional

effort, by discarding a diverging Coulomb term in the longitudinal field of the beam, the authors of Ref. [9] derived the longitudinal field for a line-charge beam ($\sigma_\perp = 0$) for the case of a bending magnet of finite length. On the other hand, the above mentioned problem of divergence for one-dimensional charge distributions, led some of the authors to consideration of ribbon beams, where the fields inside the beam are represented by convergent integrals [4, 10]. Of course, an increase of dimensionality of the problem makes it much more difficult for practical calculations.

We also note that a study of transverse forces of a line-charge beam in the case of a finite length magnet in Ref. [11] showed logarithmic singularities that cannot be not easily removed from the theory.

The existing models of line-charge beams [1, 8, 9] assume that all particles are moving through the magnetic system along the same trajectory. Strictly speaking, this kind of motion is not applicable to bunch compressors, where due to the energy chirp in the beam, particles' trajectories vary from head to tail of the beam. In addition, these models completely neglect a possible variation of the beam energy due to the converging or diverging trajectories of the beam—the effect studied in Ref. [12]—as well as a so called "compression work" [13]. It is however, remarkable, that computer simulation with ELEGANT [14] using an oversimplified model of the CSR wakefield based on approximation of Refs. [9, 15] (and neglecting the transverse CSR forces) often give results very close to those involving two- and three-dimensional CSR effects [16].

The goal of this paper is to develop a general one-dimensional description of self forces in a relativistic beam. In addition to forces due to the coherent synchrotron radiation, our formulation takes into account the space charge effect (in ultrarelativistic approximation) due to variation of the scalar potential inside the beam. We derive equations for both longitudinal and transverse directions, and show that in the limit of steady state motion they reproduce the previously published results.

In this study, we neglect the effect of metallic boundaries in the system and consider beam motion in free space. The Gaussian system of units is used throughout the paper.

LINE-CHARGE BEAM AND EQUATIONS OF MOTION

We introduce an infinitely thin beam as a collection of particles of charge e defined by a distribution function $\lambda(u)$, where u is a (scalar) parameter. A given value of this parameter is associated with a fluid element of the beam and does not change with time. The product $\lambda(u)du$ gives

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the charge within an infinitesimally small interval du, so that $\int \lambda(u)du = Q$, where Q is the beam charge, and the integration goes over all admissible values of u. The exact meaning of this parameter is not essential for what follows, and the theory is invariant with respect to the transformation from one parametrization to another. The shape of the beam and its position in space at time t are defined by the radius vector function $\boldsymbol{r}_0(t, u)$ which, for each value of u, gives a position of the corresponding elementary charge in space. The velocity of the particle $\boldsymbol{v}$ is equal $\boldsymbol{v}(t, u) = \partial \boldsymbol{r}_0(t, u)/\partial t$.

The equation of motion for a beam particle can be easily obtained in the Lorentz covariant form from the four dimensional formulation of the relativistic dynamics (see, e.g., [17], Eq. (12.36)):

$$\frac{dP^\alpha}{d\tau} = \frac{e}{mc} p_\nu \partial^\alpha A^\nu \,, \tag{1}$$

where P^α is the generalized 4-momentum, $P^\alpha = p^\alpha + eA^\alpha/c$, $p^\alpha = mc\gamma(1, \boldsymbol{\beta})$ is the 4-momentum, γ is the relativistic factor $\gamma = (1 - \beta^2)^{-1/2}$, A^α is the 4-vector potential, τ is the proper time, m is the mass of the particles in the beam, and c is the speed of light. Using the relation $\tau = t/\gamma$, the above equation can be written as

$$\frac{dP^\alpha}{dt} = e\hat{\partial}^\alpha(\phi - \boldsymbol{\beta} \cdot \boldsymbol{A}) \,, \tag{2}$$

where we assume that $\boldsymbol{\beta}$ on the right hand side is not differentiated by the operator ∂^α. To indicate the fact that $\boldsymbol{\beta}$ is immune to differentiation, here and below we use a hat ($\hat{\partial}$ and $\hat{\nabla}$) over the corresponding differentiation operator. Following Ref. [1], we introduce the function V,

$$V = \phi - \boldsymbol{\beta} \cdot \boldsymbol{A} \,, \tag{3}$$

and write the right hand side of (2) as $e\hat{\partial}^\alpha V$. Separating the time and the spatial components of Eq. (2), one finds

$$\frac{d(mc\gamma + e\phi/c)}{dt} = \frac{e}{c}\hat{\partial}_t V \,, \tag{4}$$

$$\frac{d(mc\gamma\boldsymbol{\beta} + e\boldsymbol{A}/c)}{dt} = -e\hat{\nabla} V \,. \tag{5}$$

In what follows, in most of the equations, we will use the approximation of an ultrarelativistic beam assuming that $|\boldsymbol{\beta}| = |\boldsymbol{v}|/c = 1$. However, we keep in equations of motion the relativistic factor γ, (setting $\gamma \to \infty$ in these equations does not make sense, because it results in straight motion of the particles).

One can show that the longitudinal (with respect to direction of $\boldsymbol{\beta}$) component of Eq. (5) reduces to Eq. (4), and hence can be discarded. Taking the transverse component of Eq. (5) and noting that from $|\boldsymbol{\beta}| = 1$ follows that $d\boldsymbol{\beta}/dt$ is perpendicular to $\boldsymbol{\beta}$, we obtain

$$\left(mc\gamma + \frac{e\phi}{c} - \frac{eV}{c}\right)\frac{d\boldsymbol{\beta}}{dt} = -e\left.\frac{d\boldsymbol{A}_\perp}{dct}\right|_\perp - e\hat{\nabla}_\perp V \,, \tag{6}$$

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where $\boldsymbol{A}_\perp = \boldsymbol{A} - \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \boldsymbol{A})$, and the subscript $\perp$ means taking a component perpendicular to vector $\boldsymbol{\beta}$.

Note an important subtlety in the definitions of the function V and $\boldsymbol{A}_\perp$. In addition of being functions of coordinates $\boldsymbol{r}$ and time t, they also depend on the vector $\boldsymbol{\beta}$ of the particle, for which equations of motion (4) and (5) are formulated. Since in our definition of a line-charge beam each particle is characterized by the parameter u, we have $V = V(\boldsymbol{r}, t, u)$, and $\boldsymbol{A}_\perp = \boldsymbol{A}_\perp(\boldsymbol{r}, t, u)$.

The functions V and $\boldsymbol{A}_\perp$ have a remarkable property that, together with the derivatives $\hat{\partial}_t V$, $\hat{\nabla}_\perp V$, and $d\boldsymbol{A}_\perp/dct$, they take finite values inside an infinitely thin beam, although the potentials ϕ and $\boldsymbol{A}$ (and the electric and magnetic fields) diverge there. Mathematically this means that the limits $\lim_{\boldsymbol{r}\to\boldsymbol{r}_0(t,u)} V(\boldsymbol{r}, t, u)$ and $\lim_{\boldsymbol{r}\to\boldsymbol{r}_0(t,u)} \boldsymbol{A}_\perp(\boldsymbol{r}, t, u)$ are finite. As we will show in the next section, this statement is true due to the assumption $|\boldsymbol{\beta}| = 1$.

ELECTROMAGNETIC POTENTIALS FOR A LINE-CHARGE BEAM

We start from the retarded potentials for a beam considered as a cold fluid and characterized by the charge density $\rho(\boldsymbol{r}, t)$ and the current density $\boldsymbol{j}(\boldsymbol{r}, t)$. The scalar and vector potentials generated by the beam are

$$\phi(\boldsymbol{r}, t) = \int dV' \frac{\rho(\boldsymbol{r}', t')}{|\boldsymbol{r} - \boldsymbol{r}'|} \,,$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{1}{c} \int dV' \frac{\boldsymbol{j}(\boldsymbol{r}', t')}{|\boldsymbol{r} - \boldsymbol{r}'|} \,, \tag{7}$$

where the retarded time $t' = t - |\boldsymbol{r} - \boldsymbol{r}'|/c$. The current density $\boldsymbol{j}$ in a cold beam approximation is equal to the product of the charge density and the velocity, $\boldsymbol{j}(\boldsymbol{r}, t) = \boldsymbol{v}(\boldsymbol{r}, t)\rho(\boldsymbol{r}, t)$.

As explained at the beginning of the previous section, a line-charge beam is characterized by a one-dimensional distribution function $\lambda(u)$. Its charge and current densities are

$$\rho(\boldsymbol{r}, t) = \int du\lambda(u)\delta(\boldsymbol{r} - \boldsymbol{r}_0(t, u)) \,,$$

$$\boldsymbol{j}(\boldsymbol{r}, t) = \int du\lambda(u)\boldsymbol{v}(u, t)\delta(\boldsymbol{r} - \boldsymbol{r}_0(t, u)) \,. \tag{8}$$

Substituting these equations into (7) and carrying out integration over $\boldsymbol{r}'$ we find

$$\phi(\boldsymbol{r}, t) = \int \frac{du\lambda(u)}{(1 - \boldsymbol{\beta}(u, t') \cdot \boldsymbol{n})|\boldsymbol{r} - \boldsymbol{r}_0(t', u)|} \,,$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \int \frac{du\lambda(u)\boldsymbol{\beta}(u)}{(1 - \boldsymbol{\beta}(u, t') \cdot \boldsymbol{n})|\boldsymbol{r} - \boldsymbol{r}_0(t', u)|} \,, \tag{9}$$

where $\boldsymbol{n}$ is the unit vector, $\boldsymbol{n} = (\boldsymbol{r} - \boldsymbol{r}_0(t', u))/|\boldsymbol{r} - \boldsymbol{r}_0(t', u)|$, and the retarded time $t' = t'(t, u, \boldsymbol{r})$ is now defined by the equation

$$c(t - t') = |\boldsymbol{r} - \boldsymbol{r}_0(t', u)| \,. \tag{10}$$

Note that if we set the observation point $\boldsymbol{r}$ equal to the position of a particle (characterized by u_0) in the bunch at time t, $\boldsymbol{r} = \boldsymbol{r}_0(t, u_0)$, both integrals diverge logarithmically as $u \to u_0$. However, the function V defined by (3),

$$V(\boldsymbol{r}, t, u_0) = \int \frac{du\lambda(u)(1 - \boldsymbol{\beta}(u_0, t) \cdot \boldsymbol{\beta}(u, t'))}{(1 - \boldsymbol{\beta}(u, t') \cdot \boldsymbol{n})|\boldsymbol{r} - \boldsymbol{r}_0(t', u)|}, \quad (11)$$

remains finite when $\boldsymbol{r} = \boldsymbol{r}_0(t, u_0)$, because the singularity due to the vanishing denominator is now cancelled by the numerator that also vanishes when $u \to u_0$ (note that $t' \to t$ when $u \to u_0$). It is important to emphasize here that this cancellation of the singularity occurs only in the limit $\gamma \to \infty$, when $|\boldsymbol{\beta}| = 1$. One can show that the spatial and time derivatives of V, that enter the right hand side of Eqs. (4) and (6), are also finite at $\boldsymbol{r} = \boldsymbol{r}_0(t, u_0)$. It is also easy to see that the transverse component of the vector potential

$$\boldsymbol{A}_\perp(\boldsymbol{r}, t, u_0) = \int \frac{du\lambda(u)}{(1 - \boldsymbol{\beta}(u, t') \cdot \boldsymbol{n})|\boldsymbol{r} - \boldsymbol{r}_0(t', u)|}$$
$$\times (\boldsymbol{\beta}(u, t') - \boldsymbol{\beta}(u_0, t)(\boldsymbol{\beta}(u_0, t) \cdot \boldsymbol{\beta}(u, t'))), \quad (12)$$

is finite at $\boldsymbol{r} = \boldsymbol{r}_0(t, u_0)$.

We now show how the above derived equation for V reduces to the formulae in Ref. [1] in the limit of steady state circular motion. We choose the variable u to be the distance measured from the center of the bunch and introduce the variable s as the path length along the beam trajectory. We assume that for the center of the bunch $s = 0$ at $t = 0$. The particles' velocity depends only on the position s on the trajectory, $\boldsymbol{\beta}(s)$, and the shape of the bunch does not change with time. Let us replace the integration variable u by the position of the particle s taken at the retarded time: $s(t, u, \boldsymbol{r}) = u + ct'(t, u, \boldsymbol{r})$. With some algebra, one can show that $ds = du/(1 - \boldsymbol{n} \cdot \boldsymbol{\beta}(t', u))$ which immediately gives the following expression for V:

$$V(\boldsymbol{r}, t, s_0) = \int \frac{ds\lambda(s - ct'(t, s, \boldsymbol{r}))(1 - \boldsymbol{\beta}(s_0) \cdot \boldsymbol{\beta}(s))}{|\boldsymbol{r} - \boldsymbol{r}_0(s)|}. \quad (13)$$

The advantage of this approach is that since s is the coordinate of a beam particle, the vectors $\boldsymbol{r}_0$ and $\boldsymbol{\beta}$ are functions of s only.

STEADY STATE CIRCULAR MOTION

The published results for a steady state circular motion derived in Refs. [1, 8] can be easily obtained from Eqs. (4), (6) and (13). Assuming a circular orbit of radius ρ and the bunch length $\sigma_z \ll \rho$, and following the derivation of Ref. [1] one can show that Eq. (4) reduces to

$$\frac{d(mc^2\gamma + e\phi)}{ecdt} = -\frac{2e}{3^{1/3}\rho^{2/3}} \int_{-\infty}^{\zeta} \frac{\partial_\xi \lambda(\xi)\, d\xi}{(\zeta - \xi)^{1/3}}, \quad (14)$$

where $\zeta = s - ct$ is the coordinate measured relative to a moving reference particle in the bunch.

The right hand side of this equation is usually associated with the longitudinal electric field generated by the bunch on the orbit. This is only true, however, if the potential ϕ in this equation does not change with time along particle's trajectory, $d\phi/dt = 0$. In general, the potential can vary along the orbit due to change of the transverse size of the beam caused by convergence or divergence of particles' trajectories as shown in Fig. 1—the effect studied in Ref. [12] in
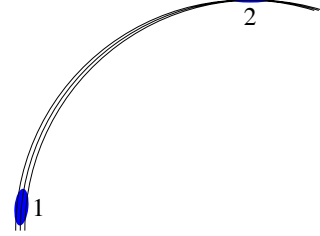

Figure 1: Due to the convergence of orbits the bunch is compressed in the transverse direction while moving on a circle from point 1 to 2.

the case of straight orbits. Eq. (14) takes a possibility such effect into account.

On the right hand side of the transverse equation of motion (6), the term containing $\boldsymbol{A}_\perp$ vanishes because the length of the vector $\boldsymbol{A}_\perp$ is constant along the particle's orbit, and hence $d\boldsymbol{A}_\perp/dt$ is perpendicular to $\boldsymbol{A}_\perp$. This means that $d\boldsymbol{A}_\perp/dt$ is directed along $\boldsymbol{\beta}$ and does not have a component perpendicular to $\boldsymbol{\beta}$, or $d\boldsymbol{A}_\perp/dt|_\perp = 0$. A direct calculation of the second term on the right hand side, in the limit $\sigma_z \ll \rho$, shows that the vector $\nabla_\perp V$ is directed toward the center of the circular orbit, and its length is

$$|\nabla_\perp V| = \frac{2\lambda(\zeta)}{\rho}. \quad (15)$$

This is the centripetal force found in Ref. [8].

INITIAL CONDITIONS FOR EQUATIONS OF MOTION

Eqs. (4) and (5) are first order differential equation with respect to time and require initial conditions specified at some initial time $t = 0$. At the same time, the functions V and $\boldsymbol{A}_\perp$, Eqs. (12) and (13), contain integration that, due to the presence of the retarded time variable t' (see Eq. (10)), extends to the region $t \to -\infty$. In practical calculations with computer codes, this contradiction is often resolved by artificially discarding the contribution from a part of the trajectory corresponding to $t < 0$. In theory, one often assumes a predefined trajectory for $-\infty < t < \infty$, and then computes the fields as a perturbation for a given motion.

TRANSITION FROM ONE CIRCULAR ORBIT TO ANOTHER AND CONCLUSION

To demonstrate usage of the formalism defined in the previous section we will show how it applies to the case

of transition from a circular orbit of radius ρ_1 to radius ρ, see Fig. 2. We assume that all particles in the bunch travel
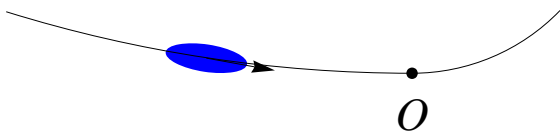

Figure 2: A bunch travelling from a circular orbit of radius ρ_1 to an orbit of radius ρ, $\rho_1 > \rho$. The two orbits are connected at point O.

along the same trajectory and the shape of the bunch does not change. Prior to entrance to the right circle, the self field of the bunch is assumed to be in steady state and described by Eqs. (14) and (15) (with ρ replaced by ρ_1).

The following parameters were assumed for numerical calculations: $\rho = 1.5$ m, $\rho_1 = 6$ m, $\sigma_z = 50$ micron. The computed right hand side of Eq. (4) is shown in Fig. 3. The quantity $\partial V / \partial t$ (normalized by $\sigma_z^{4/3} \rho^{2/3} / e$) is plotted
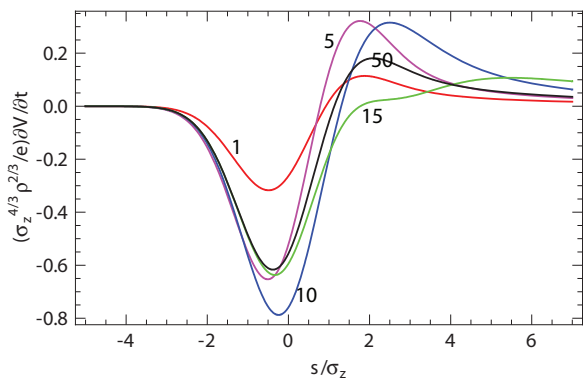

Figure 3: Normalized $\partial V / \partial t$ for several values of the distance l of the bunch from point O (indicated by a number near the curve, in cm).

as a function of distance s measured from the center of the bunch for several positions of the bunch to the right of point O (each position is characterized by distance l from O). Note that close to point O ($l = 1$ cm) the function $\partial V / \partial t$ reproduces the CSR wake for a Gaussian bunch σ_z on the circular orbit ρ_1. When l increases, the function changes and eventually (at $l = 50$ cm) approaches the CSR wake of the bunch for the circular orbit with the radius of curvature ρ (which has the same shape, but is larger by a factor of $(\rho_1/\rho)^{2/3} \approx 2.5$).

The result of numerical calculation of the right hand side of Eq. (6) is shown in Fig. 4 (where, for brevity, it is denoted by the symbol $F_\perp$ and normalized by the factor $\sigma_z \rho / e$). The curve for $l = 15$ cm approaches a steady state distribution corresponding to the radius ρ shown by the red dashed line. However, for small l, the function $F_\perp(s)$ does not approach the values that it takes when the beam is moving in the circular orbit ρ_1 (this function is shown by the blue dashed line, it is 4 times smaller than the red dashed
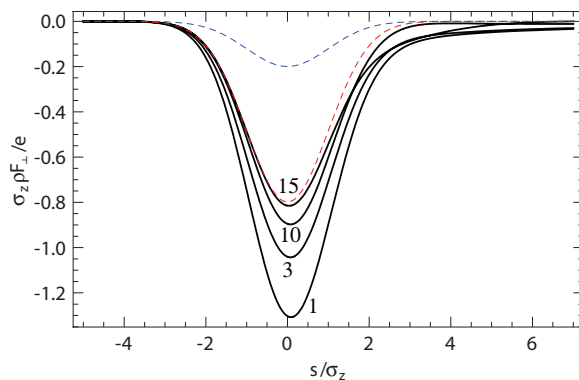
Figure 4: Normalized right had side of Eq. (6) for several values of the distance l of the bunch from point O (indicated by a number near the curve, in cm).

line). We see that the function $F_\perp(s)$ has a discontinuity when the beam passes through point O.

In conclusion, we showed that taking the limit $v \to c$ in equations of motion for a beam of charged particles, one can derive a self-consisted set of equation of motion for a line-charge distribution which takes into account electromagnetic fields generated by the beam. A remarkable property of these equations is that all quantities are finite in the limit when the transverse size of the bunch approaches zero. We demonstrated, in a simple example, how these equations describe transition fields when the curvature of the orbit abruptly changes in space.

References

[1] Ya. S. Derbenev *et al.* Report TESLA-FEL 95-05, 1995.

[2] M. Dohlus. In *EPAC 2006*, p. 1897, 2004.

[3] G. Bassi *et al.* *NIM*, **A557**, p.189, 2006.

[4] R. Talman. *PRSTAB*, **7**, p.100701, 2004.

[5] R Li. *PRSTAB*, **11**, p. 024401, 2008.

[6] R. Li and Ya. S. Derbenev. In *PAC 2005*, p. 1631, 2005.

[7] R. Talman. *PRL*, **56**, p.1429, 1986.

[8] Ya. S. Derbenev and V. D. Shiltsev. Preprint SLAC-PUB-7181, 1996.

[9] E. L. Saldin, E. A. Schneidmiller, and M. V. Yurkov. *NIM*, **A398**, p. 398, 1997.

[10] G. Bassi *et al.* In *PAC 2007*, 2007.

[11] G.A. Geloni *et al.* *NIM*, **A5722**, p.230, 2004.

[12] K. Bane and A. Chao. *PRSTAB*, **5**, p.104401, 2002.

[13] M. Dohlus. Talk at the *ICFA Beam Dynamics mini workshop*, DESY-Zeuthen, Germany, 2002.

[14] M. Borland. Report LS-287, ANL, 2000.

[15] G. Stupakov and P. Emma. In *EPAC 2002*, p. 1479, 2002.

[16] K. Bane *et al.*, these Proceedings, paper TUPPH027 2008.

[17] J. D. Jackson. *Classical Electrodynamics*. Wiley, New York, third edition, 1999.