STATISTICAL THEORY OF THE SASE FEL BASED ON THE TWO-PARTICLE CORRELATION FUNCTION EQUATION

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Abstract

The startup from noise problem in SASE FELs is usually treated in linear approximation. In this case amplification of initial density fluctuations may be calculated, and averaging over initial conditions may be fulfilled explicitly. In general nonlinear case the direct averaging is not applicable. During last years we developed the approach based on the BBGKY hierarchy for the n-particle distribution functions. The interaction of particles in FEL is retarded. Nevertheless, using special time-coordinate transformation, it is possible to eliminate the interaction lag and then to write down the BBGKY equations. Similar to plasma physics, the equations may be truncated after the second one (for the two-particle correlation function). Using this approach we consider several particular cases which illustrate some peculiar features of the SASE FEL operation.

INTRODUCTION

The short wavelength FELs are considered now as the most perspective candidates for high-brightness x-ray sources. Recent achievements in the FEL experimental technique demonstrated the capability of FELs to produce high peak brightness radiation with the wavelength down to 0.1 nm.

The short wavelength FELs usually operate in the high-gain regime and amplify initial fluctuations of electron beam current. Therefore the radiation of such FELs has stochastic nature. Its parameters may fluctuate significantly from shot to shot and within one pulse. To determine these parameters in a single shot one has to solve particle motion equations together with Maxwell equations.

As the number of particles is very large this calculation can not be fulfilled directly. There are two essentially different approaches to this problem which are used in contemporary simulation codes [1, 2]. One of them is based on the Vlasov equation for the smoothed particle distribution function in a 6-D phase space which is solved together with radiation field equation. The other one uses macroparticles to represent distribution of electrons in a beam.

For the start-up from noise simulations both approaches require special treatment of the initial conditions. In particular, macroparticle based codes have to use some artificial particle arrangement to suppress enhanced spontaneous emission [3]. There are no doubts that this method works correctly at linear stage but its applicability to the saturation stage is not so evident. In the Vlasov equation based codes, on the contrary, smoothing of the

distribution function leads to artificial damping of the initial shot noise.

Sometimes it is not necessary to know peak radiation parameters obtained in one shot. For some experiments one just needs to know the radiation parameters averaged over many shots. Moreover for long enough electron bunches averaging over the bunch length may be performed in a single shot. The averaged radiation parameters can be also used to check the single shot simulation codes. To do this one needs to average simulation results obtained from these codes for different initial conditions [4].

So it is very important to develop an adequate analytical approach and numerical algorithm for the treatment of the averaged beam and radiation parameters. This problem can be solved by the standard methods of statistical mechanics. It was previously considered by many authors but usually the solution was limited to the linear case when one can introduce the Green function and the averaging becomes straightforward.

The regular nonlinear approach to this problem was proposed in [5]. It is based on the BBGKY set of equations which is truncated to two equations for single-particle distribution function and two-particle correlation function. In this paper we give a brief overview of this approach and demonstrate its application to some particular cases.

OVERVIEW OF THE CORRELATION FUNCTION THEORY

Particle Motion Equations

To obtain the particle motion equations we make some widely-used approximations which include averaging over undulator period, paraxial solution for the radiation field equation and resonant character of the particle and field interaction. As the particle motion is paraxial, the trajectories are not perturbed significantly by radiation field. Then the motion equations for the particles with longitudinal coordinates $z^{(k)}$, relative energy deviation $\Delta^{(k)}$ and initial coordinates $X^{(k)}$ in 4-D transverse phase space may be written as

$$\frac{dz^{(k)}}{dt} = 1 - \frac{1}{2\gamma_{//}^2} + \frac{\Delta^{(k)}}{\gamma_{//}^2} - \Delta\beta\left(z^{(k)}, X^{(k)}\right)$$

$$\frac{d\Delta^{(k)}}{dt} = \sum_{l \neq i} \Phi\left[z^{(k)}, X^{(k)}, z^{(l)}\left(t'^{(l)}\right), X^{(l)}\right]$$

$$t - z^{(k)} = t'^{(l)} - z^{(l)}\left(t'^{(l)}\right) \tag{1}$$

where $\gamma_{//}$ is relativistic factor for average longitudinal velocity of the reference particle, $\Delta\beta\left(z^{(k)}, X^{(k)}\right)$ is the velocity shift due to betatron oscillations, Φ is the force acting on k-th particle from l-th particle, velocity of light is equal to unity. We assume that undulator field is zero for negative z and $\Phi\left(z^{(k)}, z^{(l)}\right)=0$ if $z^{(k)} < z^{(l)}$ or $z^{(l)} < 0$. The explicit formula for Φ can be derived from the solution of the radiation field equation [5].

It should be taken into account that (1) is not actually a system of ordinary differential equations (ODE) because its right-hand side contains particle coordinates $z^{(l)}$ at retarded moments of time $t'^{(l)} < t$. However, one can find its unique solution, if one specifies for each particle its entrance time into undulator. Though the solution procedure may be not so straightforward in general case, when one has to take into account the dependence of retarded time on particle transverse coordinate. Fortunately in this particular case the system (1) can be rewritten as the system of ODE if one introduces a new variable $\xi = t - z$. This variable can be treated simply as a new parameterization of the particle trajectories in space-time (Fig. 1). It also has obvious physical meaning. If we place the clocks along axis z and launch them using the light pulse propagating forward, then value of ξ at given point z will be equal to the "local time", which is the read-out of clock at this point. There is direct analogy between this new variable and the zone time. One can say that the clocks at each time zone are synchronized by the sunrise.

It follows from the relation $\Delta\xi = \Delta t - \Delta z$ (see Fig. 1), that for arbitrary function $f(z, t) = f(z, \xi + z)$ the derivative along the particle world line is written as

$$\frac{df}{d\xi} = \frac{df}{dt}\frac{1}{1 - V_z} \approx \frac{df}{dt} 2\gamma_{//}^2\left[1 + 2\Delta - 2\gamma_{//}^2\Delta\beta(z, X)\right] \quad (2)$$

Taking into account (2) one can write down the final system of motion equations:

$$\frac{dz^{(k)}}{d\theta} \approx \left[1 + 2\Delta^{(k)} - 2\gamma_{//}^2\Delta\beta\left(z^{(k)}, X^{(k)}\right)\right] = v_i\left(z^{(k)}, \Delta^{(k)}, X^{(k)}\right)$$

$$\frac{d\Delta^{(k)}}{d\theta} \approx \sum_{l \neq k} \Phi\left(z^{(k)}, X^{(k)}, z^{(l)}, X^{(l)}\right)$$

$$(3)$$

where $\theta = 2\gamma_{//}^2\xi$. Further we shell use $\lambda_w/2\pi = 1/k_w$ as the unit length for the longitudinal coordinates, where λ_w is the undulator period.

Initial conditions for the system (3) have to be specified at $\xi = const$. They can be reconstructed from the particle entrance time if we assume that particles move freely before they enter the undulator.
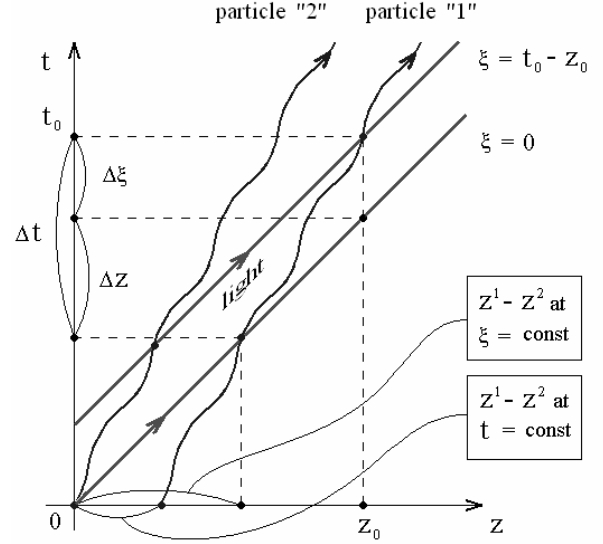

Figure 1: New parameterization of the particle world lines in space-time.

The Microscopic Density Distribution

The system (3) can not be solved directly as the number of particles is very large. But it is not required if we are interested in the results averaged over initial conditions. To make this averaging it is convenient to introduce microscopic density distribution function (Klimontovich function) in the single-particle phase plane (z, Δ)

$$N(z, \Delta, X; \theta) = \sum_k \delta\left(z - z^{(k)}(\theta)\right)\delta\left(\Delta - \Delta^{(k)}(\theta)\right)\delta\left(X - X^{(k)}\right)$$

$$(4)$$

Variable X here is a 4-D vector of initial transverse coordinates and angles. It worth noting, that X is not a dynamic variable, but just a parameter (four integrals of motion), which marks different trajectories. Nevertheless, the 6-D space (z, Δ, X) will be referred to as the phase space. The microscopic phase space density obeys continuity equation, which is equivalent to the initial system of motion equations (3)

$$\left[\frac{\partial}{\partial\theta} + \frac{\partial}{\partial z_1}v_1 + \int d\{2\}\Phi(1,2)N(2;\theta)\frac{\partial}{\partial\Delta_1}\right]N(1,\theta) = 0 \quad (5)$$

where $(i) = (z_i, \Delta_i, X_i)$, $d\{i\} = dz_i d\Delta_i dX_i$.

$N(z, \Delta, X; \theta)$ has slightly unusual physical meaning. It is a density distribution at hyperplane $\theta = const$. To describe particle dynamics in FEL one usually uses z as independent variable and $\tau_k = \frac{k_0}{k_w}\left(\frac{z}{V_{z0}} - t_k\right) = z - \theta_k$ as longitudinal coordinate. Here $k_0 \approx 2\gamma_{//}^2 k_w$ is the resonant

wave number and V_{z0} is longitudinal velocity of the reference particle. There is a simple relation between microscopic density distributions for these two different sets of variables

$$\mathrm{N}(z,\Delta,X;\theta) = \frac{1}{v(z,\Delta,X)}\mathrm{N}_z(\tau,\Delta,X;z)\Big|_{\tau=z-\theta} \qquad (6)$$

One can also easily write down a relation between $\mathrm{N}(z,\Delta,X;\theta)$ and any conventional quantity like beam current or bunching factor

$$I(z,t) = ek_0 \int v(z,\Delta,X)\mathrm{N}(z,\Delta,X;\theta)\Big|_{\theta=2\gamma_{\parallel}^2(t-z)} d\Delta dX \quad (7)$$

$$b(z) = \frac{1}{N}\int v(z,\Delta,X)\mathrm{N}(z,\Delta,X;\theta)e^{i(z-\theta)}d\Delta dX d\theta$$

where N is the total number of particles.

Averaging over Initial Conditions

To make averaging we need to introduce distribution function $f^{(N)}$ of particle coordinates in the $6N$-D phase space where N is the total number of particles. Integrating it over $N - s$ particle coordinates we can also introduce $s -$ particle distribution functions $f^{(s)}$. For the average values of the microscopic density distribution and its products we get the following expressions

$$\langle \mathrm{N}(1,\theta)\rangle = Nf^{(1)}(1,\theta) \equiv NF(1,\theta)$$

$$\langle \mathrm{N}(1,\theta)\mathrm{N}(2,\theta)\rangle = Nf^{(1)}(1,\theta)\delta(1-2) + N(N-1)\times$$
$$\times f^{(2)}(1,2,\theta)$$

The two particle correlation function G is defined by the following expression

$$f^{(2)}(1,2,\theta) = F(1,\theta)F(2,\theta) + G(1,2,\theta)$$

Averaging of (5) leads to the BBGKY chain of equations for the s – particle distribution functions. In the FEL case number of interacting particles (number of particles at cooperation length, which may be compared with the well known parameter of plasma theory – the number of plasma particles in the Debye sphere) is large and interaction has collective nature. Therefore two-particle interaction is comparatively weak. It means, that s-particle distribution functions are almost factorized (low correlations between particles). We also assume that initially particles are not correlated (N-particle distribution function is equal to the product of N equal single-particle distribution functions), so that second and higher order correlation functions are equal to zero

$$G(1,2,0) = H(1,2,3,0) = ... = 0$$

With these assumptions we can truncate the BBGKY chain to two equations for the single-particle distribution and two-particle correlation function

$$\left[\frac{\partial}{\partial\theta} + \frac{\partial}{\partial z_1}v_1\right]F(1,\theta) = -N\int\Phi(1,2)\frac{\partial}{\partial\Delta_1}G(1,2;\theta)d\{2\}$$

$$\left(\frac{\partial}{\partial\theta} + \frac{\partial}{\partial z_1}v_1 + \frac{\partial}{\partial z_2}v_2\right)G(1,2,\theta) = -N\frac{\partial F(1)}{\partial\Delta_1}\int\Phi(1,3)G(2,3,\theta)d\{3\}-$$
$$-N\frac{\partial F(2)}{\partial\Delta_2}\int\Phi(2,3)G(1,3,\theta)d\{3\} - \left(\Phi(1,2)\frac{\partial}{\partial\Delta_1} + \Phi(2,1)\frac{\partial}{\partial\Delta_2}\right)F(1)F(2)$$
$$(8)$$

Here we have taken into account oscillating nature of the interaction force Φ. It worth noting, that the second equation contains inhomogeneous part which corresponds to the shot noise. The left side of the first equation is Vlasov equation, and the right one is so-called "collision term". Eq. (8) is the equivalent of Lenard-Balescu equation in plasma physics.

Two Time Correlation Function

To determine some quantities, which can be observed in the experiment it is not sufficient to know single-time correlation function. For example the spectrum of the beam current at given point z in undulator can be found from the two-time current correlation function which is proportional to $\langle \mathrm{N}(1,\theta_1)\mathrm{N}(2,\theta_2)\rangle$, see (7). For this purpose one needs to know two-time correlation function which obeys the following equation

$$\left(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial z_1}v_1\right)G_2(1,\theta_1;2,\theta_2) = -N\frac{\partial F(1)}{\partial\Delta_1}\int\Phi(1,3)G_2(3,\theta_1;2,\theta_2)d\{3\}$$

This equation has to be solved with initial condition $G_2(1,\theta_1;2,\theta_2)\big|_{\theta_1=\theta_2} = G(1,2,\theta_1)$. It means that it can not be solved without solution of the system (8).

It worth noting, that there are some quantities which can be determined directly from the single-time correlation function, e.g. radiation peak power or angular intensity distribution.

Coasting Beam and Homogeneous Undulator

Usually electron bunch length in the short wavelength FELs is much larger than the cooperation length. Therefore one can use the coasting beam approximation which allows to reduce initial system (8) to the system of static equations without any time dependence

$$\frac{\partial}{\partial z_1}v_1 F(1) = -N\int\Phi(1,2)\frac{\partial}{\partial\Delta_1}G(1,2)d\{2\} \qquad (9)$$

$$\left(\frac{\partial}{\partial z_1}v_1 + \frac{\partial}{\partial z_2}v_2\right)G(1,2) = -N\frac{\partial F(1)}{\partial\Delta_1}\int\Phi(1,3)G(2,3)d\{3\} -$$

$$-N\frac{\partial F(2)}{\partial\Delta_2}\int\Phi(2,3)G(1,3)d\{3\} - \left(\Phi(1,2)\frac{\partial}{\partial\Delta_1} + \Phi(2,1)\frac{\partial}{\partial\Delta_2}\right)F(1)F(2)$$

$$(10)$$

In this case one usually uses different normalization of the distribution function

$$\langle F(1)\rangle = \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}\iint F(z,\Delta,X)d\Delta\,dX\,dz = 1$$

With this normalization the value of N in eq. (9)-(10) is equal to the number of particles per unit length $N = I/(ek_0)$ where I is the average beam current.

Two-time correlation function in the stationary case depends on the time difference only

$$G_2(1,\theta_1;2,\theta_2) = G_2(1,2,\theta_1-\theta_2)$$

We shell also restrict our consideration of the correlation function theory to the homogeneous undulator and matched transverse focusing case when longitudinal velocity $v_z(z,\Delta,X)$ does not depend on z explicitly (the velocity modulation due to betatron oscillations will be neglected).

Linear Theory

At linear stage one can neglect evolution of the distribution function F described by (9) and make Laplace transform of (10) over z_1 and z_2 which leads to the following integral equation

$$v(\Delta_1,X_1)s_1G(1,2) + v(\Delta_2,X_2)s_2G(1,2) +$$

$$\frac{\partial F(1)}{\partial\Delta_1}\int M(1,3)G(2,3)d3 + \frac{\partial F(2)}{\partial\Delta_2}\int M(2,3)G(1,3)d3 =$$

$$-\Phi(1,2)F(2)\frac{\partial F(1)}{\partial\Delta_1} - \Phi(2,1)F(1)\frac{\partial F(2)}{\partial\Delta_2}$$

$$(11)$$

where $(i) = (s_i,\Delta_i,X_i)$ and

$$\int M(1,3)G(2,3)d3 = \frac{N}{2\pi i}\times$$

$$\times\int_{c-i\infty}^{c+i\infty}\int\Phi(s_1,X_1,-s_3,X_3)\int G(s_2,\Delta_2,X_2,s_3,\Delta_3,X_3)d\Delta_3\,ds_3\,dX_3$$

SOLUTIONS FOR SPECIAL CASES

Cold Beam

In the simplest case of the cold beam the distribution function at eq. (11) has the following form

$$F(\Delta,X) = \tilde{F}(X)\delta(\Delta)$$

One can define the moments of correlation function by the following expression

$$g_{mn}(s_1,\mathbf{r}_1,s_2,\mathbf{r}_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\iint\Delta_1^m\Delta_2^n G(1,2)d\mathbf{r}_1'd\mathbf{r}_2'd\Delta_1 d\Delta_2$$

Then one can get from the eq. (11) the closed system of equations for g_{00} and g_{10}. For the wide beam g_{mn} depends only on $r_2 - r_1$. Making Fourier transform over this variable one can get explicit solution for g_{00}

$$g_{00}(s_1,s_2,k_\perp) = \frac{\chi}{N(s_1+s_2)}\left\{\frac{1 + 2N\chi\frac{s_1\Phi(s_1,k_\perp) + s_2\Phi(s_2,k_\perp)}{(s_1+s_2)^2}}{D(s_1,s_2,k_\perp)} - 1\right\}$$

$$(12)$$

where

$$D(s_1,s_2,k_\perp) = \left[1 + 2N\chi\frac{s_1\Phi(s_1,k_\perp) + s_2\Phi(s_2,k_\perp)}{(s_1+s_2)^2}\right]^2 -$$

$$-16(N\chi)^2\frac{s_1 s_2\Phi(s_1,k_\perp)\Phi(s_2,k_\perp)}{(s_1+s_2)^4}$$

$$(13)$$

$\Phi(s_1,k_\perp)$ is Laplace and Fourier transform of $\Phi(1,2)$ over $z_1 - z_2$ and $r_2 - r_1$, and $N\chi$ is the particle density.

The fluctuation spectrum at given point z can be determined as

$$g(z,k_z,k_\perp) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}g_{00}\left(\frac{s}{2}+ik_z,\frac{s}{2}-ik_z,k_\perp\right)e^{sz}ds \quad(14)$$

where $s = s_1 + s_2$ and $ik_z = (s_1-s_2)/2$. To calculate the integral (14) one needs to solve dispersion equation

$$D(s_1,s_2,k_\perp) = 0$$

It can be shown that this equation is equivalent to the set of two equations

$$1 + 2N\chi\frac{s_{1,2}\Phi(s_{1,2},k_\perp)}{(s_{1,2}\mp i\omega)^2} = 0$$

Each of them is the dispersion equation for the linearized single-particle Vlasov equation. They describe a perturbation with frequency ω. Elimination of ω will return us to the dispersion equation $D = 0$. Taking into

account the explicit expression for $\Phi(s_1, k_\perp)$ one gets the following set of equations

$$\left[s_{1,2}{}^2 + (1-\alpha^2)^2 \right] \left(s_{1,2} \mp i\omega \right)^2 - 16\rho^3 s_{1,2}^2 = 0 \qquad (15)$$

where $\rho \ll 1$ is the Pierce parameter and $\alpha^2 = \dfrac{k_\perp^2}{2k_0 k_w}$.

Eq. (15) can be solved by perturbation method. The solution with maximum $\mathrm{Re}(s)$ has the flowing form

$$s = s_1 + s_2 \approx 2\sqrt{3}\rho \left(1 - \frac{1}{36} \left[\tilde{v} + \frac{\alpha^2}{\rho} \right]^2 \right) \qquad (16)$$

where $v = \rho\tilde{v} = k_z - 1$. One can see from (16) that the gain length $L_G = \dfrac{1}{2\sqrt{3}\rho k_w}$ and the characteristic amplification bandwidth at one gain length $\sigma_v = 3\sqrt{2}\rho$ obtained in this theory are the same as in the conventional 1-D theory of SASE FEL [6].

Evolution of the transversal coherence is described by inverse Fourier transform of $g(z, k_z, k_\perp)$

$$g(\tilde{z}, \tilde{v}, \tilde{r}) \sim e^{\tilde{z}} \int e^{-\frac{\tilde{z}}{36}\left(\tilde{v} + \frac{\sqrt{3}}{2\pi} k_\perp^2 \right)^2} J_0(k_\perp \tilde{r}) k_\perp dk_\perp \qquad (16)$$

Here $\tilde{z}$ is normalized to L_G and $\tilde{r}$ is normalized to $\sqrt{L_G \lambda_0}$, $J_0(x)$ is the Bessel function. Behaviour of (16) as a function of $\tilde{z}$ and $\tilde{r}$ for $\tilde{v} = 0$ is illustrated by Fig. 2.
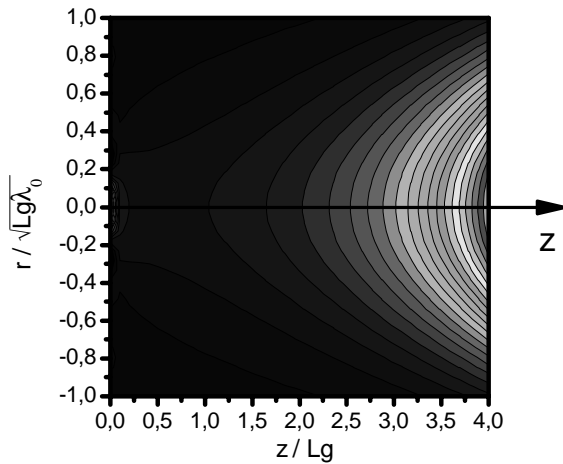

Figure 2: Evolution of transversal coherence in the case of wide beam.

Numerical Solution for the 1-D case and Comparison with Quasilinear Approach

There is no analytical solution of (9)-(10) for the saturation stage but one can relatively easily obtain the numerical solution of these equations for the 1-D case. Though it is not very interesting from the practical point of view one can use the results obtained from these simulations to check the validity of the results obtained from single shot simulation codes. In this section we present the results of such comparison of the correlation function approach with the simulation code based on the quasilinear equations [7].

In the quasilinear approach one solves the set of equations for the smoothed density distribution in single-particle phase space with conventional coordinates (τ, Δ) and z as independent variable. We assume this distribution to be periodic in τ

$$f(\tau, \Delta, z) = F_0(\Delta) + \sum_v f_v(\Delta, z) e^{i(1+v)\tau} \qquad (17)$$

where $v = H_v n$ and $T = 2\pi/H_v$ is the period in τ which has to be chosen larger then several tens of cooperation lengths. In the quasilinear approximation one has the following set of equations

$$\frac{\partial F_0}{\partial z} = 2\,\mathrm{Re}\left(\sum_v \frac{\partial f_v}{\partial \Delta} \overline{A_v} \right)$$

$$\frac{\partial f_v}{\partial z} + i(2\Delta - v) f_v = A_v \frac{\partial F_0}{\partial \Delta}$$

$$\frac{dA_v}{dz} = 4\rho^3 \int f_v(\Delta, z) d\Delta$$

In the second equation terms, proportional to the radiation field A_v and components of distribution function, which oscillate at double frequency, are neglected. This simplification is called "quasilinear approximation". Its physical meaning is simple. There are two mechanisms, which stop the exponential signal growth. One corresponds to almost full beam bunching. It works at low initial velocity spread, for example, in travelling wave tubes. For short-wavelength FELs the longitudinal velocity spread is relatively high, so, before the higher harmonics of density modulation may appear, the energy spread increases, increasing the gain length. This second mechanism is described by the quasilinear equations. To describe SASE, these equations have to be solved with the shot noise initial load. The initial distribution function for this case can be obtained the following way. One has to start with the initial microscopic density distribution

$$f_0(\tau,\Delta)=\frac{T}{N_T}\sum_{i=1}^{N_T}\delta(\tau-\tau_i)\delta(\Delta-\Delta_i) \qquad (18)$$

where N_T is the number of particles per one period. Making Fourier transform of (18) one obtains the following expression for $f_{0\nu}(\Delta)$

$$f_{0\nu}(\Delta)=\frac{1}{T}\int_0^T f_0(\tau,\Delta)e^{-i(1+\nu)\tau}d\tau=\frac{1}{N_T}\sum_{k=1}^{N_T}\delta(\Delta-\Delta_k)e^{-i\varphi_k}$$

where $\varphi_k=(1+\nu)\tau_k$ are random phases with uniform distribution. In simulation delta functions have to be replaced by step functions

$$f_{0\nu}(\Delta)=\frac{1}{N_T}\frac{1}{H_\Delta}\sum_m\Pi(\Delta-\Delta_m)\sum_{k=1}^{N_{\Delta_m}}e^{-i\varphi_k}$$

where $\Pi(\Delta)=1$ if $\Delta\in(-0.5H_\Delta,0.5H_\Delta)$ and $\Pi(\Delta)=0$ otherwise, $N_{\Delta_m}\approx N_T F_0(\Delta_m)H_\Delta$ is number of particles in the m-th energy interval. Taking into account that for the sum of large number of random phasors $\sum_{k=1}^N e^{i\varphi_k}=X+iY$ one has the following distribution

$$F(X,Y)dXdY\approx\frac{1}{N\pi}e^{-\frac{1}{N}(X^2+Y^2)}dXdY=\frac{d\phi}{2\pi}\frac{1}{N}e^{-\frac{R^2}{N}}dR^2$$

one can get the final expression for $f_{0\nu}(\Delta_m)$

$$f_{0\nu}(\Delta_m)=\frac{1}{\sqrt{N_\lambda}}\sqrt{\frac{H_\nu}{H_\Delta}}\sqrt{F_0(\Delta_m)}\sqrt{-\ln(I_1)}e^{2\pi iI_2}$$

where H_ν is the step in ν, $I_{1,2}$ are random numbers distributed uniformly in the interval $(0,1)$.

To make comparison of two approaches one needs to find correspondence between them. Taking into account (6) and (17) one can write down the following relation

$$G_2(z_1,\Delta_1,z_2,\Delta_2,\theta_1-\theta_2)\approx$$
$$\approx\sum_\nu\langle f_\nu(z_1,\Delta_1)f_\nu^*(z_2,\Delta_2)\rangle e^{i(1+\nu)[(z_1-z_2)-(\theta_1-\theta_2)]}$$

where angle brackets mean averaging over initial conditions. From this relation one can determine two quantities which can be compared for two different approaches

$$J_0(z)=\int G(z,\Delta_1,z,\Delta_2)d\Delta_1 d\Delta_2 \approx$$
$$\approx\int\sum_\nu\langle f_\nu(z,\Delta_1)f_\nu^*(z,\Delta_2)\rangle d\Delta_1 d\Delta_2$$

$$J_\nu(z)=\int G_2(z,\Delta_1,z,\Delta_2,\tau)e^{i(1+\nu)\tau}d\Delta_1 d\Delta_2 d\tau \approx$$
$$\approx T\langle f_\nu(z,\Delta_1)f_\nu^*(z,\Delta_2)\rangle$$

We made the comparison for the following set of parameters: Pierce parameter $\rho=5\cdot10^{-4}$, energy spread $\sigma_\Delta=0.5\rho$, number of particles per one wavelength $N_{\lambda_0}=10^4$. Description of the numerical algorithm can be found at [8]. Averaging in the quasilinear simulation was done over 1000 runs. The spectrum distribution for single run is shown at Fig. 3. The comparison results are presented at Fig. 4-6.
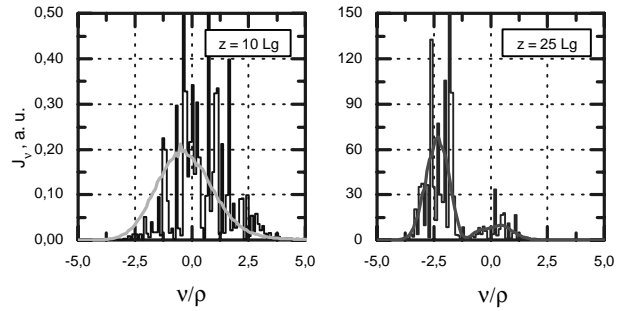

Figure 3: Beam current spectra obtained in quasilinear simulations. Step function – single run, smooth curve – averaged over 1000 runs.
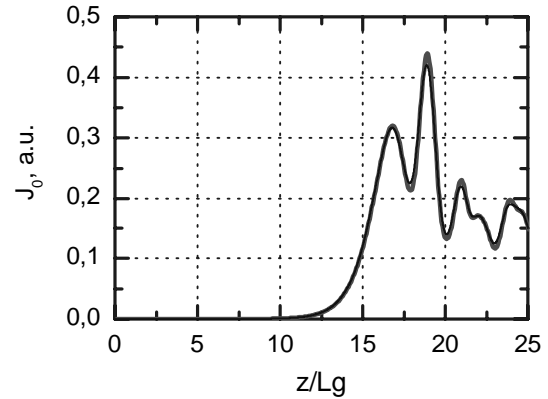

Figure 4: Dependence of the current fluctuation power on the longitudinal coordinate in undulator. Thick red curve – correlation function approach, thin blue curve – quasilinear approach, L_g - the gain length at the linear stage.
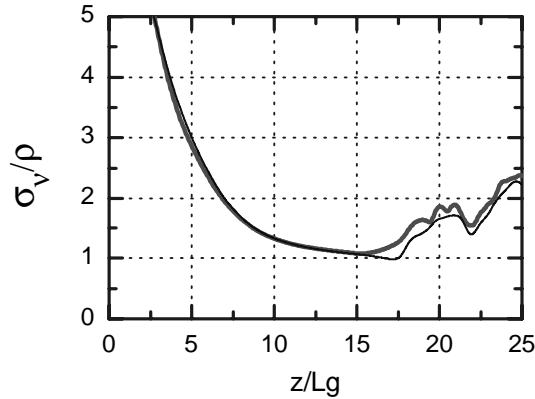
Figure 5: Dependence of the r.m.s. spectral bandwidth on the longitudinal coordinate in undulator. Thick red curve – correlation function approach, thin blue curve – quasilinear approach.

CONCLUSION

In this paper we reviewed the correlation function approach to the theory of SASE FELs. New analytical and numerical results were described. Cross-checking with the quasilinear theory approach gives excellent mutual agreement. The correlation function approach is significant for both right understanding of noise in FEL and obtaining of reliable calculation results. At this point only simplified model problems were solved this way, but, the codes for real FEL calculations also looks feasible.

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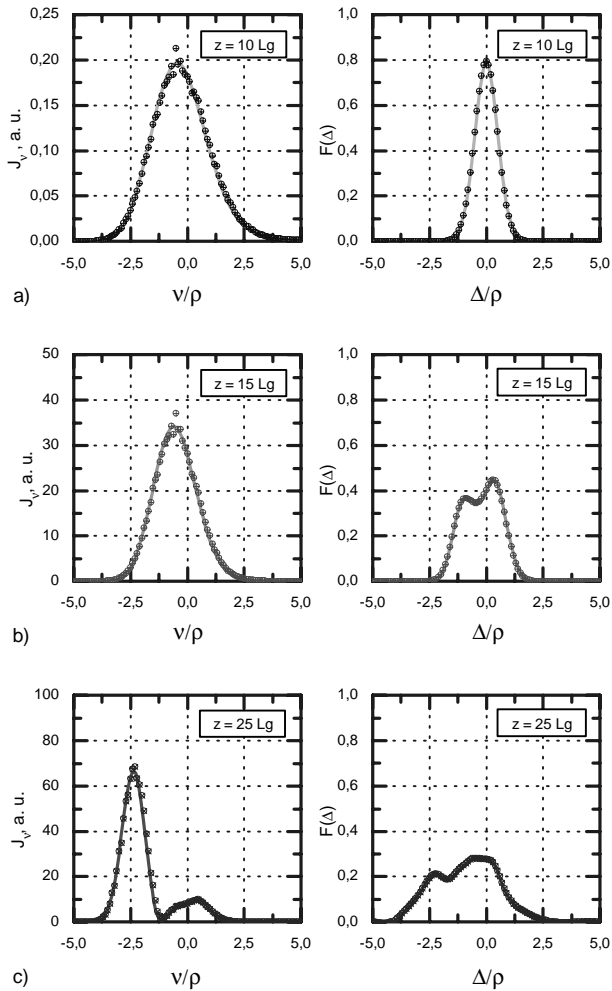[8] O.A. Shevchenko, N.A. Vinokurov, Nucl. Instr. and Meth. A 603 (2009) 46

Figure 6: The spectral and energy distributions at different longitudinal positions. Solid curve – correlation function simulations, separate dots – quasilinear simulations.