MODAL DESCRIPTION OF LONGITUDINAL SPACE-CHARGE FIELDS IN PULSE-DRIVEN FREE-ELECTRON DEVICES *

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Abstract

Longitudinal space-charge fields taking place in free-electron devices are considered in the framework of a three-dimensional, space-frequency approach. The model is based on the expansion of the total electromagnetic field (radiation and space-charge waves) in terms of transverse eigen-modes of the (cold) cavity, in which the field is excited and propagates. Using the model, an analytical expression for longitudinal electric field is derived for a point-like charge, moving along a waveguide with a constant velocity. This enables comparison of different components of the resulted longitudinal electric field, including contributions of forward and backward going waves, of near and under cut-off frequencies, and others.

INTRODUCTION

Development of experimental technic of electron bunch compression and of laser induced photocathodes enables application of ultra short high intensive electron pulses in free-electron radiation sources. In free-electron lasers (FELs), a highly efficient super-radiant emission can be achieved with such pulses, providing a strong coherent radiation with intensity being proportional to the pulse charge squared [1]. Unfortunately, longitudinal space-charge fields give rise to expansion of short electron pulses along their trajectory, restricting or limiting applications of intense ultra-short electron pulses [2, 3].

In the present work, longitudinal space-charge fields considered in the framework of three-dimensional, space-frequency approach [4, 5]. The model is based on the expansion of the total electromagnetic field (radiation and space-charge waves) in terms of transverse eigenmodes of the (cold) cavity, in which the field is excited and propagates. This approach has been successfully applied for the analysis of wide-band interactions in free-electron lasers operating in the linear and non-linear regimes [6]-[9].

The basic equations of the model, originally obtained as a solution of the wave equations for the electromagnetic field in an uniform waveguide, are shown to satisfy also Gauss's law for electric and magnetic fields. Longitudinal electric field was found in the model analytically for a point-like charge, moving along a waveguide with a constant velocity. This enables consideration and comparison of different components of the resulted longitudinal electric field, such as contributions of forward and backward going waves, of near and under cut-off frequencies, and others.

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MODAL PRESENTATION OF ELECTROMAGNETIC FIELD

In the approach, the total electromagnetic field in the positive-frequency domain is expanded in terms of a complete set of transverse eigenmodes of the medium in which the field is excited and propagates:

$$\begin{aligned} \widetilde{\mathbf{E}}_\perp(\mathbf{r},f) &= \sum_{\pm q} C_q(z,f)\, e^{-jk_{zq}z}\, \widetilde{\mathcal{E}}_{q\perp}(\mathbf{r}_\perp) \\ \widetilde{\mathbf{H}}_\perp(\mathbf{r},f) &= \sum_{\pm q} C_q(z,f)\, e^{-jk_{zq}z}\, \widetilde{\mathcal{H}}_{q\perp}(\mathbf{r}_\perp) \end{aligned} \quad (1)$$

where tilde symbols a positive frequency domain quantity, $C_q(z,f)$ is scalar amplitude of the qth mode (the summation includes both forward and backward modes) with electric field $\widetilde{\mathcal{E}}_{q\perp}(\mathbf{r}_\perp)$ and magnetic field $\widetilde{\mathcal{H}}_{q\perp}(\mathbf{r}_\perp)$ profiles, and axial wavenumber is:

$$k_{zq}(f) = \begin{cases} \sqrt{k^2 - k_{\perp q}^2}\,, & k > k_{\perp q} \text{ (prop.)} \\ -j\sqrt{k_{\perp q}^2 - k^2}\,, & k < k_{\perp q} \text{ (cut-off)} \end{cases} \quad (2)$$

here $k = w/c$ and $c = 1/\sqrt{\epsilon\mu}$ is the velocity of light.

Expressions for the longitudinal component of the electric and magnetic fields are obtained after substituting the modal representation (1) of the fields into Faraday and Ampere equations, where source of electric current density $\widetilde{\mathbf{J}}(\mathbf{r},f)$ is introduced:

$$\begin{aligned} \widetilde{E}_z(\mathbf{r},f) &= \sum_{\pm q} C_q(z,f)\, e^{-jk_{zq}z}\, \widetilde{\mathcal{E}}_{qz}(\mathbf{r}_\perp) \\ &\qquad - \frac{1}{jw\epsilon}\widetilde{J}_z(\mathbf{r},f) \\ \widetilde{H}_z(\mathbf{r},f) &= \sum_{\pm q} C_q(z,f)\, e^{-jk_{zq}z}\, \widetilde{\mathcal{H}}_{qz}(\mathbf{r}_\perp) \end{aligned} \quad (3)$$

Evolution of the amplitudes of the excited modes is described by a set of coupled first order differential equations:

$$\frac{d}{dz}C_{\pm q}(z,f) = \mp\frac{1}{2\mathcal{N}_q}\, e^{\pm jk_{zq}z} \times \\ \times \int\!\!\int \left[\left(\frac{Z_q}{Z_q^*}\right)\widetilde{\mathbf{J}}_\perp(\mathbf{r},f) \pm \hat{\mathbf{z}}\widetilde{J}_z(\mathbf{r},f)\right] \cdot \widetilde{\mathcal{E}}_q^*(\mathbf{r}_\perp)\, d\mathbf{r}_\perp \quad (4)$$

Field amplitudes of each mode is normalized via a complex Poynting vector power:

$$\mathcal{N}_q = \int\!\!\int \left[\widetilde{\mathcal{E}}_{q\perp}(\mathbf{r}_\perp) \times \widetilde{\mathcal{H}}_{q\perp}^*(\mathbf{r}_\perp)\right] \cdot \hat{\mathbf{z}}\, d\mathbf{r}_\perp \quad (5)$$

and the mode impedance is given by:

$$Z_q = \begin{cases} \sqrt{\frac{\mu}{\epsilon}}\frac{k}{k_{zq}} = \frac{w\mu}{k_{zq}} & \text{for TE modes} \\ \sqrt{\frac{\mu}{\epsilon}}\frac{k_{zq}}{k} = \frac{k_{zq}}{w\epsilon} & \text{for TM modes} \end{cases} \quad (6)$$

Electromagnetic field given by equations (1), (3), (4) is found as solutions of the Faraday and Ampere law's of Maxwell's equations in the frequency domain. As easy to show, divergence of the magnetic field is

$$\begin{aligned}\nabla\cdot\widetilde{\mathbf{H}}(\mathbf{r},f) &= \nabla_\perp\cdot\widetilde{\mathbf{H}}_\perp(\mathbf{r},f)+\frac{\partial}{\partial z}\widetilde{\mathbf{H}}_z(\mathbf{r},f)\\ &= -\sum_{+q}\frac{<\widetilde{\mathcal{E}}_{q_z}|\widetilde{\mathbf{J}}_z>}{\mathcal{N}_q}\widetilde{\mathcal{H}}_{q_z}(\mathbf{r}_\perp)\end{aligned} \quad (7)$$

where

$$<\widetilde{\mathcal{E}}_{q_z}|\widetilde{\mathbf{J}}_z> \equiv \int\!\!\int \widetilde{\mathbf{J}}_z(\mathbf{r};f)\cdot\widetilde{\mathcal{E}}^*_{q_z}(\mathbf{r}_\perp)\,\mathrm{d}\mathbf{r}_\perp \quad (8)$$

Since $\widetilde{\mathcal{E}}_{q_z}(x,y)=0$ for TE modes and $\widetilde{\mathcal{H}}_{q_z}(x,y)=0$ for TM modes, the Gauss's law for magnetic fields

$$\nabla\cdot\mathbf{H}(\mathbf{r},t)=0 \quad (9)$$

is also satisfied.

Divergence of the electric field can be expressed as follows:

$$\begin{aligned}\nabla\cdot\widetilde{\mathbf{E}}(\mathbf{r},f) &= \nabla_\perp\cdot\widetilde{\mathbf{E}}_\perp(\mathbf{r},f)+\frac{\partial}{\partial z}\widetilde{\mathbf{E}}_z(\mathbf{r},f)=\\ &= -\sum_{+q} i_q(z)\,\widetilde{\mathcal{E}}_{q_z}(\mathbf{r}_\perp)-\frac{1}{jw\epsilon}\frac{\partial}{\partial z}\widetilde{J}_z(\mathbf{r},f)\end{aligned} \quad (10)$$

where

$$i_q(z)\equiv\frac{1}{\mathcal{N}_q}\left(\frac{Z_q}{Z_q^*}\right)\int\!\!\int\widetilde{\mathbf{J}}_\perp(\mathbf{r};f)\cdot\widetilde{\mathcal{E}}^*_{q_\perp}(\mathbf{r}_\perp)\,\mathrm{d}\mathbf{r}_\perp \quad (11)$$

The first term in (10) can be rewritten:

$$\begin{aligned}-\sum_{+q} i_q(z)\,\widetilde{\mathcal{E}}_{q_z}(\mathbf{r}_\perp) &= -\frac{1}{jk_{zq}}\nabla_\perp\cdot\sum_{+q} i_q(z)\,\widetilde{\mathcal{E}}_{q_\perp}(\mathbf{r}_\perp)=\\ &= -\frac{1}{jw\epsilon}\,\nabla_\perp\cdot\widetilde{\mathbf{J}}_\perp(\mathbf{r};f)\end{aligned} \quad (12)$$

Therefore

$$\nabla\cdot\widetilde{\mathbf{E}}(\mathbf{r},f)=-\frac{1}{jw\epsilon}\,\nabla\cdot\widetilde{\mathbf{J}}(\mathbf{r};f)=\frac{\widetilde{\rho}(\mathbf{r};f)}{\epsilon} \quad (13)$$

where $\widetilde{\rho}(\mathbf{r};f)$ is the charge density in the positive-frequency domain and the continuity equation

$$\nabla\cdot\widetilde{\mathbf{J}}(\mathbf{r};f)=-jw\,\widetilde{\rho}(\mathbf{r};f) \quad (14)$$

is applied.

The last term in Eq. (3) introduces the one-dimensional (1D) longitudinal space-charge field. In the time domain, the term corresponds to the field given by:

$$\begin{aligned}\Delta E_z(\mathbf{r},t) &= -\Re\left\{\int_0^\infty\frac{1}{jw\epsilon}\widetilde{J}_z(\mathbf{r},f)\,e^{+jwt}\,\mathrm{d}f\right\}\\ &= \frac{\sigma_>(\mathbf{r})-\sigma_<(\mathbf{r})}{2\epsilon}\end{aligned} \quad (15)$$

where

$$\sigma_>(\mathbf{r},t)\equiv\int_t^{+\infty}J_z(\mathbf{r},\tau)\,\mathrm{d}\tau \qquad \sigma_<(\mathbf{r},t)\equiv\int_{-\infty}^{t}J_z(\mathbf{r},\tau)\,\mathrm{d}\tau \quad (16)$$

are the corresponding charge densities of the unit square.

According to (3), longitudinal electric field is given as a superposition of 1D longitudinal space-charge field term (15) and the summation of all TM modes. As will be demonstrated, 1D longitudinal space-charge field is sufficiently compensated in such a superposition. The problem, however, is that only a very few first modes are usually considered in real simulations, and TM modes are often are not taken into account at all. To formulate equation (3) for longitudinal electric field in a more balanced way, transverse dependence of longitudinal current density can be expanded in terms of the same complete set of eigenmodes $\widetilde{\mathcal{E}}_{q_z}(\mathbf{r}_\perp)$:

$$\widetilde{\mathbf{J}}_z(\mathbf{r};f)=\sum_{+q}\frac{w\epsilon k_{z_q}}{k^2_{\perp_q}}\mathcal{N}_q<\widetilde{\mathcal{E}}_{q_z}|\widetilde{\mathbf{J}}_z>\widetilde{\mathcal{E}}_{q_z}(\mathbf{r}_\perp) \quad (17)$$

here the normalization of TM modes is used. Substituting the expansion into (3), presentation for the longitudinal electric field can be re-written:

$$\begin{aligned}\widetilde{E}_z(\mathbf{r},f) &=\\ &\sum_q\Big[C_{+q}(z,f)\,e^{-jk_{zq}z}-C_{-q}(z,f)\,e^{+jk_{zq}z}\\ &\quad -\frac{k_{z_q}}{jk^2_{\perp_q}}\mathcal{N}_q<\widetilde{\mathcal{E}}_{q_z}|\widetilde{\mathbf{J}}_z>\Big]\,\widetilde{\mathcal{E}}_{qz}(\mathbf{r}_\perp)\end{aligned} \quad (18)$$

THE ELECTRON BEAM DYNAMICS

The state of a particle i is described by a six-components vector, which consists of the particle's position coordinates $\mathbf{r}_i=(\mathbf{r}_{\perp i},z_i)$ and velocity vector $\mathbf{v}_i$. The velocity of each particle, in the presence of electromagnetic field, is found from the Lorentz force equation:

$$\frac{d\mathbf{v}_i}{dz}=-\frac{e}{m}\frac{\mathbf{E}(\mathbf{r}_i,t_i)+\mathbf{v}_i\times\mathbf{B}(\mathbf{r}_i,t_i)}{\gamma_i\,v_{z_i}}-\frac{\mathbf{v}_i}{\gamma_i}\frac{d\gamma_i}{dz} \quad (19)$$

where e and m are the electron charge and mass respectively, and the Lorentz relativistic factor γ_i of each particle is found from the equation for kinetic energy:

$$\frac{d\gamma_i}{dz}=-\frac{e}{mc^2}\frac{1}{v_{z_i}}\,\mathbf{v}_i\cdot\mathbf{E}(\mathbf{r}_i,t_i) \quad (20)$$

The fields in equation (19) represent the total (DC and AC) forces operating on the particle, and include also the self-field due to space-charge.

The current distribution is determined by the position and the velocity of the particles in the beam:

$$\mathbf{J}(\mathbf{r},t)=-\frac{Q}{N}\sum_{i=1}^{N}\frac{\mathbf{v}_i}{v_{z_i}}\,\delta(\mathbf{r}_\perp-\mathbf{r}_{\perp i})\,\delta\left[t-t_i(z)\right] \quad (21)$$

Here $Q = I_0 T$ is the total charge of the e-beam pulse with DC current I_0 and temporal duration T, and $t_i(z)$ is the time it takes a particle to arrive at a position z.

In the positive frequency domain, current density of the drive beam is given by:

$$\widetilde{\mathbf{J}}(\mathbf{r},f) = 2\int_{-\infty}^{+\infty} \mathbf{J}(\mathbf{r},t)\, e^{+jwt}\, dt$$

$$= -2\frac{Q}{N}\sum_{i=1}^{N} \frac{\mathbf{v}_i}{v_{z_i}}\,\delta(\mathbf{r}_\perp - \mathbf{r}_{\perp i})\, e^{+jwt_i(z)} \quad (22)$$

and the excitation equation (4) can be rewritten as follows:

$$\frac{d}{dz}C_{\pm q}(z,f) =$$

$$\pm\frac{1}{\mathcal{N}_q}\frac{Q}{N}\sum_{i=1}^{N}\left\{\frac{Z_q}{Z_q^*}\frac{\mathbf{v}_{\perp_p}\cdot\widetilde{\mathcal{E}}^*_{\pm q_\perp}(\mathbf{r}_{\perp i})}{v_{z_i}} + \widetilde{\mathcal{E}}^*_{\pm q_z}(\mathbf{r}_{\perp i})\right\}\times$$

$$\times\ \exp\left(j[wt_i(z) \mp k_{z_q} z]\right) \quad (23)$$

The resulted expression together with the beam dynamics equations (19), (20) form a close set of equations, enabling a self-consistent solution of the electromagnetic fields (radiation and space-charge waves) in electron devices and free-electron lasers.

SINGLE CHARGE

To check applicability of equation (18) to calculations of longitudinal electric space-charge fields, lets consider a point-like charge, moving along a waveguide of length L with a constant velocity v_z. In this case, the field amplitudes $C_{\pm q}(z,f)$ can be found analytically:

$$\begin{aligned} C_{+q}(z,f) &= \frac{Q}{\mathcal{N}_q}\widetilde{\mathcal{E}}^*_q(\mathbf{r}_{\perp p})\frac{1-e^{-j\theta_{+q}z}}{j\theta_{+q}} \\ C_{-q}(z,f) &= \frac{Q}{\mathcal{N}_q}\widetilde{\mathcal{E}}^*_q(\mathbf{r}_{\perp p})\frac{e^{-j\theta_{-q}L}-e^{-j\theta_{-q}z}}{j\theta_{-q}} \end{aligned} \quad (24)$$

here $\mathbf{r}_{\perp p}$ is transverse position of the particle, and $\theta_{\pm q} = \frac{w}{v_z} \mp k_{zq}$. Figure 1 demonstrates the frequency dependence of this field amplitudes excited by a point 1 nC charge, moving with 5.5 MeV energy along a rectangular 15×10 mm^2 waveguide of the length of $L = 10$ cm. The field is checked at the moment when the charge overs the point $z = L/2$. Both forward and backward going waves are found to play a comparative role in the calculations, dominating in the vicinity of cut-off and at zero frequency, so that no one of them can be neglected.

Field amplitudes (24) results in the following analytic expression for the longitudinal electric field of the charge:

$$E_z(\mathbf{r},t) = \sum_{TM_{+q}}\left\{E^{(pr)}_{zq}(\mathbf{r},t) + E^{(co)}_{zq}(\mathbf{r},t) + \Delta E_{zq}(\mathbf{r},t)\right\} \quad (25)$$

where

$$E^{(pr)}_{zq}(\mathbf{r},t) = \frac{Qk^2_{\perp q}}{2\pi\epsilon}\frac{\widetilde{\mathcal{E}}^*_{qz}(\mathbf{r}_{\perp p})\widetilde{\mathcal{E}}_{qz}(\mathbf{r}_\perp)}{<\widetilde{\mathcal{E}}_{qz}|\widetilde{\mathcal{E}}_{qz}>}\times$$

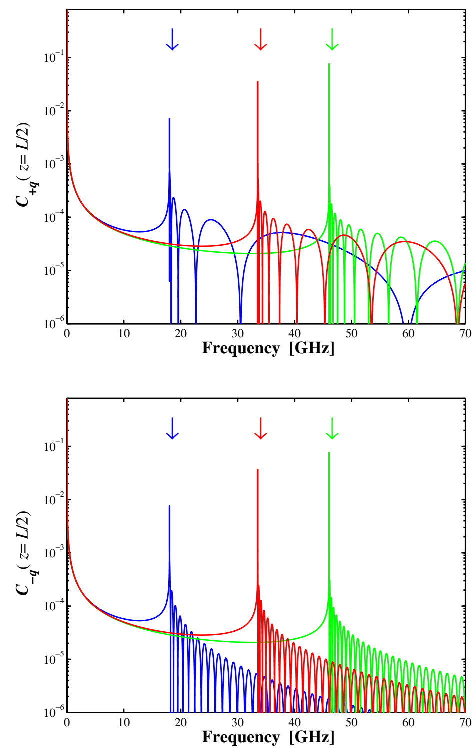


Figure 1: Forward (top) and backward (bottom) wave field amplitudes (24). Arrows show the cut-off frequencies of the first 3 modes taken into account.

$$\times\left\{\int_{ck_{\perp q}}^{\infty}\frac{\sin(wt - k_{zq}z)}{(w/v_z)^2 - k^2_{zq}}\left(\frac{1}{k_{zq}v_z} - \frac{1}{w}\right)dw\right.$$

$$-\int_{ck_{\perp q}}^{\infty}\frac{\sin(w[t-L/v_z] - k_{zq}[L-z])}{(w/v_z)^2 - k^2_{zq}}\left(\frac{1}{k_{zq}v_z} - \frac{1}{w}\right)dw$$

$$\left.-2\int_{ck_{\perp q}}^{\infty}\frac{\sin(w[t-z/v_z])}{(w/v_z)^2 - k^2_{zq}}\frac{dw}{w}\right\} \quad (26)$$

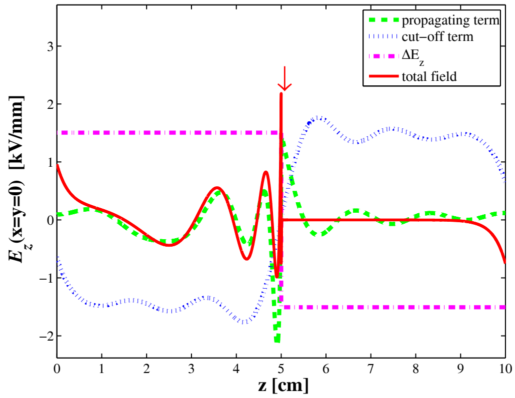
is the field component corresponding to propagating waves,

$$E^{(co)}_{zq}(\mathbf{r},t) = \frac{Qk^2_{\perp q}}{2\pi\epsilon}\frac{\widetilde{\mathcal{E}}^*_{qz}(\mathbf{r}_{\perp p})\widetilde{\mathcal{E}}_{qz}(\mathbf{r}_\perp)}{<\widetilde{\mathcal{E}}_{qz}|\widetilde{\mathcal{E}}_{qz}>}\times$$

$$\times\left\{\int_{0}^{ck_{\perp q}}\frac{e^{-|k_{zq}|z}}{(w/v_z)^2 + |k_{zq}|^2}\left[\frac{\cos(wt)}{|k_{zq}|v_z} - \frac{\sin(wt)}{w}\right]dw\right.$$

$$-\int_{0}^{ck_{\perp q}}\frac{e^{-|k_{zq}|(L-z)}}{(w/v_z)^2 + |k_{zq}|^2}$$

$$\cdot\left[\frac{\cos(w[t-L/v_z])}{|k_{zq}|v_z} - \frac{\sin(w[t-L/v_z])}{w}\right]dw$$



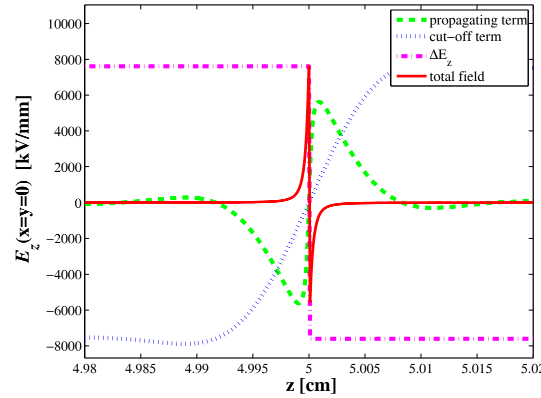


Figure 2: Longitudinal electric field (25) obtained with a single TM_{11} mode (left) and with 5050 $TM_{m,n}$ modes (right).

$$-2\int_0^{ck_{\perp q}} \frac{\sin(w[t-z/v_z])}{(w/v_z)^2+|k_{zq}|^2}\frac{dw}{w}\Bigg\} \tag{27}$$

corresponds to cut-off waves, and

$$\Delta E_{zq}(\mathbf{r},t) = \frac{Q}{2\epsilon}\frac{\widetilde{\mathcal{E}}^*_{qz}(\mathbf{r}_{\perp p})\widetilde{\mathcal{E}}_{qz}(\mathbf{r}_\perp)}{<\widetilde{\mathcal{E}}_{qz}|\widetilde{\mathcal{E}}_{qz}>}\,\mathrm{sgn}(t-z/v_z) \tag{28}$$

presents 1D longitudinal field term.

Figure 2 demonstrates the field (25) found at the waveguide axis at the moment when the charge over the point $z = L/2$. The calculations were carried out with a single TM_{11} mode and with 5050 $TM_{m,n}$ modes ($m+n \le 200$), respectively. As seen from the pictures, step-like 1D longitudinal field term (15) is totally compensated at long distances from the charge, mainly with the field component originated from cut-off waves, resulting in a short-range field. At closer distances, propagating waves are playing an important role too. Convergence of the calculations is found to be extremely slow and is demonstrated in the figure 3. Lorenz transformed (to the Lab system)

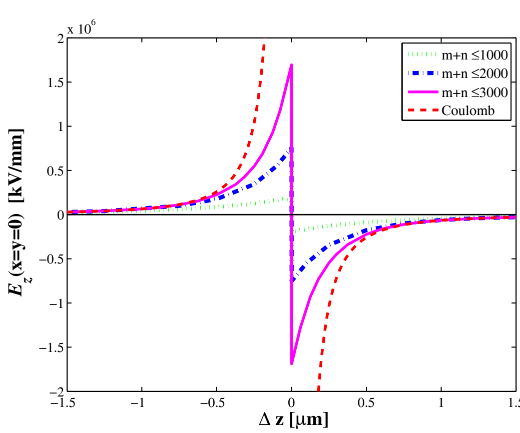


Figure 3: Longitudinal electric field (25).

Coulomb field of the point charge is given too for comparison. Note, that Coulomb field considered here does not fulfill the boundary conditions at the waveguide walls; however it can be close to a correct solution if the waveguide axis is only considered.

CONCLUSIONS

Space-frequency model is found to satisfy the full set of Maxwell's equations, including Gauss's law for electric and magnetic fields. Expressions describing longitudinal electric field in the model include, in addition to a simple step-like 1D term, summation over both forward and backward going waves, which contribute mainly at zero frequency and near cut-off. No one of these terms can be neglected in the expressions. Calculations of longitudinal electric field of a point-like charge according to Eq. (3) or (18) demonstrate an extremely slow convergence. To improve the efficiency of the calculations, a simple 3D approximation for space-charge fields should be applied.

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