

NUMERICAL SIMULATION OF KOLMOGOROV ENTROPY IN A FREE-ELECTRON LASER WITH ION-CHANNEL GUIDING

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Abstract

The dynamical stability of electron trajectories in a free-electron laser with planar wiggler is studied. The analysis is based on the numerical simulation of Kolmogorov entropy to investigate how the separation of the trajectories of two neighbouring electrons in the six-dimensional phase space evolves along the undulator. Self-electric and self-magnetic fields are taken into account and an adiabatically tapered wiggler magnetic field is used in order to inject the electrons into the wiggler. A considerable decrease in the dynamical stability of electron trajectories was found near the resonance region. It was found that self-fields decrease the dynamical stability of electron trajectories in group I orbits and increase it in group II orbits. Furthermore, the electromagnetic radiation weakens the dynamical stability of electrons as it grows exponentially and become very intense near the saturation point.

INTRODUCTION

The free-electron laser (FEL) in the millimetre or submillimetre regime operates with a high density and low energy relativistic electron beam. In this operating regime, the self-electric and self-magnetic fields of the electron beam are strong and a focusing mechanism, like an axial magnetic field, is necessary to confine the beam. It has been suggested that an electron beam may be confined by passing it through an ion channel, which may provide an alternative to the use of an axial magnetic field. It has been found by several theoretical investigations that ion-channel guiding may provide some advantages [1-3].

The chaotic motion of an electron in FEL, has been studied using Poincaré surface-of-section method, in helical [4-10] as well as planar [8, 11] wiggler. In these studies axial magnetic field guiding has been used whereas helical wiggler with ion-channel guiding was used in [10].

In some of the above studies [6-8, 10, 11] computation of Liapunov exponents has been used in a limited manner to confirm the results of chaoticity. Zhang and Elgin¹² used Kolmogorov entropy (Liapunov exponents) to study the self-field effects on the dynamical stability of an electron motion in a FEL with planar wiggler. But their results are invalid because they did not use a focusing mechanism like an axial magnetic field to confine the beam. In their analysis, the electron under the influence of self-fields, will drift away from the axis and diverge long

before reaching the end of the wiggler. The same thing happens if the adiabatic wiggler magnetic field is removed; therefore, their conclusion in Ref. 12 that the adiabatic entry field is trivial in 1D wiggler is incorrect.

The purpose of the present paper is to study the dynamical stability of an electron motion, using Kolmogorov entropy, in a FEL with planar wiggler when the focusing of the beam is provided by an ion channel. The latter problem has not been studied to the best of our knowledge.

KOLMOGOROV ENTROPY

Our analysis is based on the numerical solution concerning Kolmogorov or metric entropy therefore we will describe this approach briefly in this section. In studying chaotic motions it is important to see if the motion is sensitive to small changes in initial conditions [13]. Usually, one should expect closely neighbored trajectories to diverge exponentially in time for chaotic motion and to separate only linearly in time for the regular motion. Consider a reference electron with initial conditions $\mathbf{x}_r(t=0) = \mathbf{x}_r(0)$ and the trajectory $\mathbf{x}_r = (x_{1r}, x_{2r}, \dots, x_{6r})$ in the 6D phase space. Now choose a closely neighbored electron with initial conditions $\mathbf{x}_n(t=0) = \mathbf{x}_r(0) + \delta\mathbf{x}(0)$ and the trajectory $\mathbf{x}_n = (x_{1n}, x_{2n}, \dots, x_{6n})$, with $\delta\mathbf{x}(0) \ll \mathbf{x}_r(0)$. To study the separation of these two electrons with time we define a quantity

$$k = \lim_{\substack{t \rightarrow \infty \\ |d_0| \rightarrow 0}} \left(\frac{1}{t}\right) \ln \left(\frac{|d(t)|}{|d_0|}\right), \quad (1)$$

where $|d(t)|$ is the norm or the separation of the two trajectories in the phase space

$$|d(t)| = \sqrt{\sum_{j=1}^6 (x_{jn} - x_{jr})^2}, \quad (2)$$

and $|d_0| = |d(t=0)|$. k is zero for the regular motion because $|d(t)|$ grows linearly with time and it grows exponentially for the chaotic motion and k becomes a positive number. For negative k , $|d(t)|$ converges exponentially. Therefore, k determines the dynamical stability of the motion which is related to the chaoticity of the motion.

Two problems may cause difficulties in computation of k . One is that if the norm $|d(t)|$ increases exponentially

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we might encounter computational overflow. The second one is that in a bounded system the computational time is not infinite. Benettin et al. [14] proposed the following procedure, related to the Kolmogorov entropy, to solve these problems. The total time T is divided into N equal and small interval τ so that $T = N\tau$. In this procedure, the trajectory of the neighbouring electron is not computed continuously. Rather, at the beginning of each time step the norm $|d(t)|$ is brought back to the original norm at $|d_0|$. This scheme may be iterated from $t = 0$ to $t = \tau$ in N steps to obtain a sequence of positive numbers $|d_i|$, $i = 1, 2, \dots, N$ and calculate the following quantity;

$$k_N(\tau, x, d) = \frac{1}{\tau N} \sum_{i=1}^N \ln \frac{|d_i|}{|d_0|}. \quad (3)$$

Benettin et al. [14] have proved that the limit of $k_N(N \rightarrow \infty)$ exists and this limit is independent of $|d_0|$. Furthermore, if the initial point $\bar{x}_r(0)$ is chosen to be close to a chaotic region of phase space, then $k = \lim_{N \rightarrow \infty} k_N$ will be independent of the initial condition $\bar{x}_r(0)$. k is called the entropy-like quantity and is related to the Kolmogorov entropy. The value of k shows how the trajectories of two closely spaced electrons in phase space diverge from each other. A small value of k indicates a small separation of the trajectories. When k is positive the motion is chaotic and the closely neighbored trajectories separate exponentially. On the other hand, when k is negative the motion is nonchaotic and the neighbouring trajectories approach each other. It should be noted that since the interaction length of FEL systems are finite we should take the iteration step very small and the number of iteration very large, e.g. of the order of 10^8 , so that the condition ($k = \lim_{N \rightarrow \infty} k_N$) is approximately met.

EQUATION OF MOTION

The relativistic equations of motion of a single electron in external electromagnetic fields \mathbf{E} and \mathbf{B} are

$$\frac{d(\gamma m_0 \mathbf{v})}{dt} = -e(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}). \quad (4)$$

By introducing dimensionless velocity

$$\bar{\mathbf{V}} = \gamma \frac{\mathbf{v}}{c} \text{ and } \gamma = \sqrt{1 + \bar{\mathbf{V}}^2}, \quad (5)$$

equation of motion (4) can be written as

$$\frac{d\bar{\mathbf{V}}}{dt} = \frac{-e}{m_0 c} (\mathbf{E} + \frac{1}{\sqrt{1 + \bar{\mathbf{V}}^2}} \bar{\mathbf{V}} \times \mathbf{B}). \quad (6)$$

In order to write Eq. (6) in a dimensionless form we introduce normalized quantities in Gaussian units.

$$\bar{\mathbf{E}} = \frac{e}{m_0 c^2 k_w} \mathbf{E}, \quad (7)$$

$$\bar{\mathbf{B}} = \frac{e}{m_0 c^2 k_w} \mathbf{B}, \quad (8)$$

$$\bar{x}_i = k_w x_i, \bar{t} = c k_w t, \quad (9)$$

to obtain

$$\frac{d\bar{V}_x}{d\bar{z}} = -\frac{1}{\bar{V}_z} [\sqrt{1 + \bar{\mathbf{V}}^2} \bar{E}_x + \bar{V}_y \bar{B}_z - \bar{V}_z \bar{B}_y], \quad (10)$$

$$\frac{d\bar{V}_y}{d\bar{z}} = -\frac{1}{\bar{V}_z} [\sqrt{1 + \bar{\mathbf{V}}^2} \bar{E}_y + \bar{V}_z \bar{B}_x - \bar{V}_x \bar{B}_z], \quad (11)$$

$$\frac{d\bar{V}_z}{d\bar{z}} = -\frac{1}{\bar{V}_z} [\sqrt{1 + \bar{\mathbf{V}}^2} \bar{E}_z + \bar{V}_x \bar{B}_y - \bar{V}_y \bar{B}_x]. \quad (12)$$

Here, the independent variable has been changed from t to z using $d/dt = v_z(d/dz)$.

Equations (10)-(12) form a set of first order ordinary differential equations that will be solved numerically, in this analysis, by fourth order Runge-Kutta method to study the dynamical stability of a single electron in a FEL with an ion channel under influence of self-fields and a laser field. The iteration step is taken equal to $0.01 \mu\text{m}$ which corresponds to 3×10^8 total number of iteration for an interaction length of 3 m. The initial position of the reference electron is chosen to be on the radial position of 0.0065 mm and zero azimuthal position. The relativistic factor is 3. The neighbouring electron has the same position with slightly different relativistic factor equal to $3 \times (1 + 10^{-8})$. This corresponds to an initial Euclidean norm of 3.18×10^{-8} . Initial velocities of both electrons are in the axial direction.

There are two methods to solve Eqs. (10)-(12). In the first method we directly solve the set of equations (10)-(12) for both the reference and the neighbouring electron. In the second method instead of solving these equations for the neighbouring electron we expand and linearize Eqs. (10)-(12) around coordinates of the reference electron and along the tangent vector $\delta\bar{x}_i = \bar{x}_{ni} - \bar{x}_{ri}$, to find differential equations for $\delta\bar{x}_i$ [13]. Equations (10)-(12) are represented by

$$\frac{d\delta\bar{x}_i}{d\bar{z}} = G_i(\bar{x}, \bar{y}, \dots, \bar{V}_z), \quad i = 1, 2, \dots, 6 \quad (13)$$

Now we linearize the equations about the coordinates of the reference electron $\bar{\mathbf{x}}_r = (\bar{x}_r, \bar{y}_r, \dots, \bar{V}_{zr})$ to yield the tangent map

$$\frac{d\delta\bar{x}_i}{d\bar{z}} = \sum_{j=1}^6 \delta\bar{x}_j \left(\frac{\partial G_i}{\partial \bar{x}_j} \right)_{\bar{x}=\bar{x}(\bar{z})}. \quad (14)$$

Solving these equations numerically will yield $\delta\bar{x}_i$ to obtain the norm

$$|d(\bar{z})| = \sqrt{\sum_{i=1}^6 \delta\bar{x}_i^2}. \quad (15)$$

In the present analysis we use the linearized tangent method, however, the direct method yields almost identical results for the final entropy-like quantity k_N .

Parameters of the FEL used in the numerical calculation are as follows. The amplitude of the wiggler field is taken as $B_w = 800 G$ and its wavelength as

$\lambda_w = 2$ cm. The initial power is assumed to be 1 W and its FEL gain length is equal to $l_g = 18$ cm. The density of the electron beam is $n_b = 1.7 \times 10^{12}$ cm⁻³ which corresponds to the current $I = 1000$ A and the radius of the beam $R_b = 0.2$ cm. The radiation wavelength is approximately found from $\lambda_l = \lambda_w / 2\gamma_z^2$ mm.

In many occasions self-fields of a relativistic electron beam are important to be considered and they may be written as¹⁵

$$\mathbf{E}_s = -\frac{m_0 \omega_p^2}{2e} (x \hat{e}_x + y \hat{e}_y), \quad (16)$$

$$\mathbf{B}_s = \frac{m_0 v_{z0} \omega_p^2}{2ec} (y \hat{e}_x - x \hat{e}_y), \quad (17)$$

where $\omega_p = (4\pi e^2 n_b / m_0)^{1/2}$ is the plasma frequency.

PLANAR WIGGLER

The planar wiggler magnetic field can be written as

$$\mathbf{B}_w = -\hat{e}_y B_w(z) \sin k_w z. \quad (18)$$

Since initial conditions with longitudinal velocity correspond only to the points prior to entrance into the interaction region, an adiabatic entry field of the wiggler is necessary in order to inject the electron into the equilibrium and steady-state trajectories. It should be noted that in the absence of adiabatic entry wiggler field, under the above mentioned initial conditions, electrons will drift away from the axis and will hit the drift tube wall. The adiabatic magnetic field is

$$B_w(z) = \begin{cases} B_w \sin \frac{k_w z}{4N_w}, & 0 \leq k_w z \leq 2\pi N_w \\ B_w, & 2\pi N_w \leq k_w z \end{cases} \quad (19)$$

where N_w is the period number of the adiabatic magnetic field.

In the linear phase of a FEL, laser field grows exponentially and for a plane polarized radiation may be written as

$$\mathbf{E}_l = \hat{e}_x E_0 e^{z/l_g} \cos(k_l z - \omega_l t), \quad (20)$$

$$\mathbf{B}_l = \hat{e}_y c^{-1} E_0 e^{z/l_g} \cos(k_l z - \omega_l t), \quad (21)$$

An expression for the radiation power can be obtained from Eqs. (26) and (27),

$$P = \frac{\pi r_b^2}{c \mu_0} E_0^2 \exp\left(\frac{2z}{l_g}\right) \cos^2(k_l z - \omega_l t), \quad (22)$$

where $R_b = 0.2$ cm. In the numerical analysis the initial power is assumed to be 1W, which corresponds to $\bar{E}_0 = 3.41 \times 10^{-5}$.

Ion channel

The transverse electrostatic field generated by an ion channel may be expressed by

$$\mathbf{E}_i = 2\pi e n_i (x \hat{e}_x + y \hat{e}_y). \quad (23)$$

The numerical and theoretical analysis of the relativistic motion of an electron in a FEL with planar wiggler and ion-channel guiding with or without self-fields and in the absence of radiation show that the equilibrium orbits, similar to the case of axial magnetic field, divide into group I and group II orbits. Figure 1 shows equilibrium orbits, by numerical computation, with self-fields neglected.

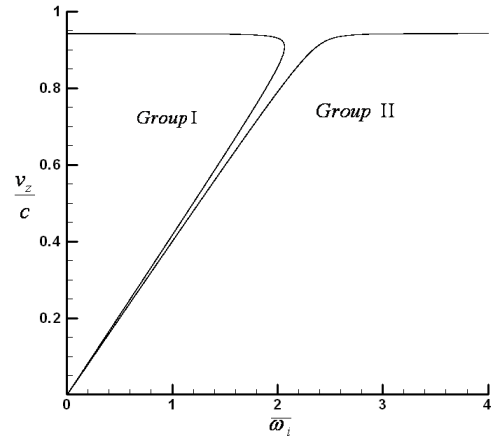


Figure 1: Equilibrium trajectories with ion-channel guiding and with self-fields neglected

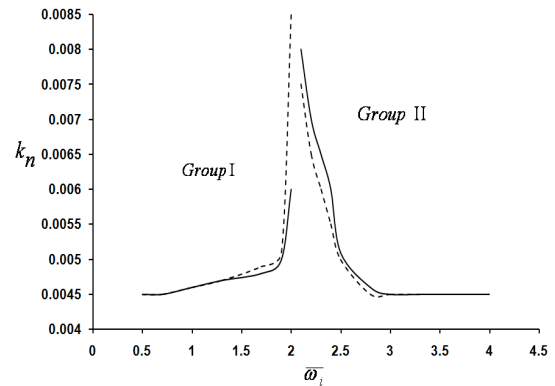


Figure 2: Variation of Kolmogorov entropy k_N with $\bar{\omega}_i$. Self-fields are included in solid line but they are neglected in dotted lines.

Figure 2 shows how the entropy-like quantity k_N varies with ion-channel density, represented by the normalized ion-channel frequency $\bar{\omega}_i = (4\pi n_i e^2 / mc^2 k_w^2)^{1/2}$, in the absence of radiation. Self-fields are neglected in dotted lines and they are included in solid lines. Sharp increase of Kolmogorov entropy k_N in the resonance region indicates stronger dynamical instability of the electron motion, in both group I and group II orbits, compared to other regions whether or not self-fields are included in the analysis. Figure 2 also shows that self-fields have negligible effects

on k_N away from the resonance region for both group I and group II orbits. Self-fields increase the dynamical stability in group I orbits and decrease it in group II orbits.

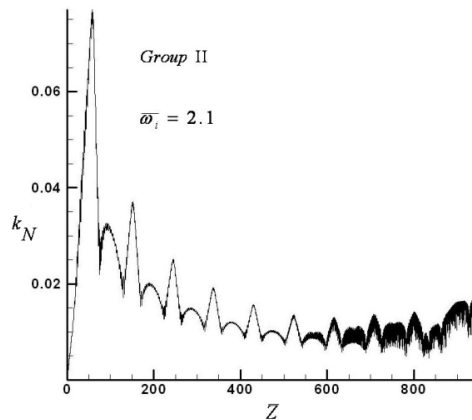


Figure 3: Evolution of Kolmogorov entropy k_N with \bar{z} when self-fields and exponentially growing radiation are included.

Figure 3 shows the evolution of Kolmogorov entropy k_N with \bar{z} when self-fields and radiation are included in the computation and the focusing of the beam is made by ion channel. Similar to the case of axial magnetic field the laser field deteriorates the stability. As the laser field grows exponentially in FEL, dynamical stability deteriorates mainly at the end of the undulator, where the amplitude of radiation becomes very large near the saturation point.

CONCLUSION

In this analysis the dynamical stability of a relativistic electron motion in a FEL with a planar wiggler is studied by numerical solution of Kolmogorov entropy (entropy-like quantity). The self fields of the electron beam and exponentially growing radiation is taken into account and focusing of the beam is made by an ion channel.

For this focusing mechanism, sharp increase of k_N in the resonance region is found with or without the self-field effects, which is indicative of a relative dynamical

instability of the resonance region. It was found that self-fields increase the dynamical stability in group I orbits and decrease it in group II orbits. It is also shown that presence of a strong radiation in the FEL deteriorates the dynamical stability. Contrary to what is reported in Ref. 12, we have shown that an injection mechanism, like an adiabatically tapered wiggler, and a focusing mechanism like an ion channel is necessary. Otherwise, injected electrons will drift away from the z-axis and diverge.

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