MULTISTAGE BUNCH COMPRESSION

Igor Zagorodnov and Martin Dohlus
DESY, Hamburg, Germany

Abstract

The nonlinearities of the RF fields and the dispersion sections can be corrected with a higher harmonic RF module. In this paper we present an analytical solution for nonlinearity correction up to the third order in a *multistage* bunch compression and acceleration system without collective effects. A more general solution for a system with collective effects (space charge, wakefields, CSR effects) can be found by iterative tracking procedure based on this analytical result. We apply the developed formalism to study two stage bunch compression in FLASH (see a companion paper [1]). Analytical estimations of RF tolerances are given.

INTRODUCTION

Free-electron lasers require an electron beam with high peak current and low transverse emittance. In order to meet these requirements several bunch compressors are usually planned in the beam line [2], [3].

The nonlinearities of the radio frequency (RF) fields and of the bunch compressors (BC's) can be corrected with a higher harmonic RF system [4]. An analytical solution for cancellation of RF and BC's nonlinearities for a one stage bunch compressor system was presented in [4]. The second order treatment of multistage bunch compressor systems was done in [5], where the difficulty to extend the third-order analysis to multistage systems was pointed out as well.

In this paper we present, for the first time, an analytical solution for the nonlinearity correction up to the third order in a *multistage* bunch compression and acceleration system without collective effects for an arbitrary number of stages. A more general solution for a system with collective effects (space charge forces, wakefields, a coherent synchrotron radiation (CSR) within the chicane magnets) can be found by an iterative tracking procedure based on this analytical result. We apply the developed formalism to study the two stage bunch compression scheme at FLASH [2] (see a companion paper [1]). The analytical estimations of RF tolerances are given for two and three stage bunch compression as well.

ANALYTICAL SOLUTION OF MULTISTAGE BUNCH COMPRESSION PROBLEM WITHOUT SELF-FIELDS

Problem Formulation

Let us consider the transformation of the longitudinal phase space distribution in a multistage bunch compression and accelerating system shown in Fig.1. The system has N bunch compressors ($\mathrm{BC}_1,\ldots,\mathrm{BC}_N$) and N accelerating modules ($\mathrm{M}_1,\ldots,\mathrm{M}_N$). The first module consists of the fundamental harmonic module $\mathrm{M}_{1,1}$ and of the higher harmonic module $\mathrm{M}_{1,n}$ placed as shown in Fig. 1.

The energy changes in accelerating modules M_i, $\mathrm{M}_{1,1}$ can be approximated as

$$\Delta E_{1,1}(s) = V_{1,1}\cos(ks+\varphi_{1,1}),$$

$$\Delta E_i(s) = V_i\cos(ks_{i-1}(s)+\varphi_i),\ i>1,$$

where φ_i is a phase, V_i is the on crest voltage and k is a wave number.

The energy change in the high harmonic module is given by

$$\Delta E_{1,n}(s) = V_{1,n}\cos(nks+\varphi_{1,n}).$$

The relative energy deviations in bunch compressors read

$$\delta_1(s) = \frac{(1+\delta(s))E_0^0+\Delta E_{1,1}(s)+\Delta E_{1,n}(s)}{E_1^0}-1,$$

$$\delta_i(s) = \frac{(1+\delta_{i-1}(s))E_{i-1}^0+\Delta E_i(s)}{E_i^0}-1, \qquad i=2,\ldots,N.$$

The transformation of the longitudinal coordinate in compressor BC_i can be approximated by the expression

$$s_i(s) = s_{i-1}(s)-\left(r_{56i}\delta_i(s)+t_{56i}\delta_i^2(s)+u_{56i}\delta_i^3(s)\right),$$
$$i=1,\ldots,N,$$

where we have used a simplified notation ($r_{56i}\equiv R_{56}^{(i)}$, $t_{56i}\equiv T_{566}^{(i)}, u_{56i}\equiv U_{5666}^{(i)}$, see [4]) for the momentum compaction factors in compressor number i.

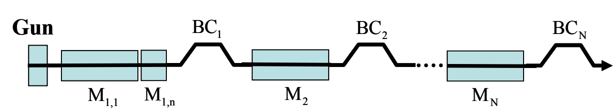

Figure 1: The multistage bunch compression system with the high harmonic module at the first stage.

In order to simplify the notation in the equations below we introduce a new function $Z_i(s)\equiv s_i'(s)$ and the *inverse* bunch compression factors

$$Z_i\equiv s_i'(0),\ Z_i'\equiv s_i''(0),\ Z_i''\equiv s_i'''(0).$$

Let us suggest that we know the desired energies $\{E_i^0\}$ and the desired compression factors $\{Z_i^0\}$ in all bunch compressors. For the linear compression in the middle of the bunch we would like to have the first and the second derivatives of the global compression equal to zero: $Z_N'=0$, $Z_N''=0$. In general case they could take arbitrary values $Z_N'^0$ and $Z_N''^0$.

In order to find $2N+2$ settings of RF parameters $V_{1,1}$, $\varphi_{1,1}$, $V_{1,n}$, $\varphi_{1,n}$, $\{V_i,\varphi_i\}$, $i=2,3,...,N$, of the accelerating modules we have to solve the non-linear system of $2N+2$ equations

$$\begin{cases} \delta_i(0)=0, \quad i=1,...,N, \\ s_i'(0)=Z_i^0, \quad i=1,...,N, \\ s_N''(0)=Z_N'^0, \quad s_N'''(0)=Z_N''^0. \end{cases} \tag{1}$$

Analytical Solution of the Multistage Bunch Compression Problem

In order to simplify the form of the solution and to generalize it for arbitrary number of stages we split system (1) in two independent problems.

To simplify the notation let us introduce the new variables

$$X_{1,n}+iY_{1,n}=V_{1,n}e^{i\varphi_{1,n}}, \quad X_{1,1}+iY_{1,1}=V_{1,1}e^{i\varphi_{1,1}},$$

$$X_i+iY_i=V_ie^{i\varphi_i}, \; i>1,$$

$$\mathbf{X}=\left(X_2,...,X_N\right)^T, \qquad \mathbf{Y}=\left(Y_2,...,Y_N\right)^T .$$

Then the first problem for $2N+1$ variables reads

$$\begin{cases} \delta_i(0,\mathbf{X})=0, \quad i=2,...,N, \\ s_i'(0,\mathbf{X},\mathbf{Y},\alpha_1)=Z_i^0, \quad i=1,...,N, \\ s_N''(0,\mathbf{X},\mathbf{Y},\alpha_1,\alpha_2)=Z_N'^0, \\ s_N'''(0,\mathbf{X},\mathbf{Y},\boldsymbol{\alpha})=Z_N''^0, \end{cases} \tag{2}$$

where $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)^T$, $\alpha_i=\dfrac{\partial^i\delta_1}{\partial s^i}(0)$, is an unknown vector which describes up to the third order the energy curve immediately after the high harmonic module. If we know the solution of system (2) then we can formulate the second problem for the RF parameters in module M_1.

The second problem for 4 variables reads

$$\begin{cases} \delta_1(0,X_{1,1},Y_{1,1},X_{1,n},Y_{1,n})=0, \\ \delta_1'(0,X_{1,1},Y_{1,1},X_{1,n},Y_{1,n})=\alpha_1, \\ \delta_1''(0,X_{1,1},Y_{1,1},X_{1,n},Y_{1,n})=\alpha_2, \\ \delta_1'''(0,X_{1,1},Y_{1,1},X_{1,n},Y_{1,n})=\alpha_3, \end{cases} \tag{3}$$

The last problem can be rewritten as a linear system

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -k & 0 & -nk \\ -k^2 & 0 & -(nk)^2 & 0 \\ 0 & k^3 & 0 & (nk)^3 \end{pmatrix} \begin{pmatrix} X_{1,1} \\ Y_{1,1} \\ X_{1,n} \\ Y_{1,n} \end{pmatrix} = \begin{pmatrix} E_1^0-E_0^0 \\ E_1^0\alpha_1-E_0^0\delta_0'(0) \\ E_1^0\alpha_2-E_0^0\delta_0''(0) \\ E_1^0\alpha_1-E_0^0\delta_0'''(0) \end{pmatrix}. \tag{4}$$

If the initial values E_0^0, $\delta_0'(0)$, $\delta_0''(0)$, $\delta_0'''(0)$ and the variables α_i, $i=1,2,3$, are known then the solution of Eq. (4) reads

$$X_{1,1}=\frac{F_3+F_1(kn)^2}{k^2(n^2-1)}, \; Y_{1,1}=-\frac{F_4+F_2(kn)^2}{k^3(n^2-1)}, \tag{5}$$

$$X_{1,n}=-\frac{F_3+F_1k^2}{k^2(n^2-1)}, \; Y_{1,n}=\frac{F_4+F_2k^2}{k^3n(n^2-1)},$$

where

$$F_1=E_1^0-E_0^0, \quad F_i=E_1^0\alpha_{i-1}-E_0^0\frac{\partial^{i-1}\delta_0}{\partial s^{i-1}}(0), \; i=2,3,4 .$$

The main difficulty which remains is to find the solution of non-linear system (2). In order to write explicitly the last two equations in system (2) we need to find the first three derivatives of functions $s_i(s)$ and $\delta_i(s)$. In the following we omit argument s. In this simplified notation the first three derivatives at $s=0$ read

$$s_i'=s_{i-1}'-r_{56i}\delta_i', \quad s_i''=s_{i-1}''-r_{56i}\delta_i''-2t_{56i}\left(\delta_i'\right)^2, \tag{6}$$

$$s_i'''=s_{i-1}'''-r_{56i}\delta_i'''-6t_{56i}\delta_i'\delta_i''-6u_{56i}\left(\delta_i'\right)^3, \qquad i=1,...,N,$$

$$s_0'\equiv1, \; s_0''\equiv0, \; s_0'''\equiv0,$$

$$\delta_i'=\frac{\delta_{i-1}'E_{i-1}^0-kZ_{i-1}Y_i}{E_i^0} \;\; \delta_i''=\frac{\delta_{i-1}''E_{i-1}^0-k^2Z_{i-1}^2X_i-kZ_{i-1}'Y_i}{E_i^0},$$

$$\delta_i'''=\frac{\delta_{i-1}'''E_{i-1}^0-k^3Z_{i-1}^3Y_i-3k^2Z_{i-1}Z_{i-1}'X_i-kZ_{i-1}''Y_i}{E_i^0},$$

$$\delta_1'\equiv\alpha_1, \; \delta_1''\equiv\alpha_2, \; \delta_1'''\equiv\alpha_3 .$$

Let us describe the solution of system (2) step by step. At the beginning, from the first N equations, $\delta_i(0,\mathbf{X})=0$, we can easily find the components of vector $\mathbf{X}$:

$$X_i=E_i^0-E_{i-1}^0, \; i=2,...,N . \tag{7}$$

From the next $N+1$ equations, $s_i'(0,\mathbf{X},\mathbf{Y},\alpha_1)=Z_i^0$, $i=1,...,N$, we find the components of vector $\mathbf{Y}$ and the energy chirp $\alpha_1\equiv\delta_1'$ before BC_1:

$$\delta_i'=\frac{Z_{i-1}-Z_i}{r_{56i}}, \; i=1,...,N, \tag{8}$$

$$Y_i=\frac{\delta_{i-1}'E_{i-1}^0-\delta_i'E_i^0}{kZ_{i-1}}, \quad i=2,...,N . \tag{9}$$

From equation $s_N''(0,\mathbf{X},\mathbf{Y},\alpha_1,\alpha_2)=Z_N'^0$ we can find parameter α_2. This equation can be rewritten as a system of linear difference equations (see Eqs. (5), (6))

$$\begin{cases} x_i=x_{i-1}+a_iy_i+b_i, \quad i=1,...,N, \\ y_i=y_{i-1}+d_ix_{i-1}+e_i, \quad i=2,...,N, \\ x_0=0, \quad x_N=x_N^0, \end{cases} \tag{10}$$

where

$$x_i\equiv s_i'', \; y_i\equiv E_i^0\delta_i'', \; x_N^0\equiv Z_N'^0,$$

$$a_i=-\frac{r_{56i}}{E_i^0}, \; b_i=-2t_{56i}\left(\delta_i'\right)^2, \; i=1,...,N,$$

$$d_i=-kY_i, \; e_i=-k^2Z_{i-1}^2X_i, \; i=2,...,N .$$

It is easy to check that the solution of the problem (10) can be found as

$$\alpha_2=\frac{y_1}{E_1^0}, \; y_1=\frac{Z_N'^0-\tilde{x}_N}{\bar{x}_N}, \tag{11}$$

where $\bar{x}_N$ and $\tilde{x}_N$ are solutions of the particular homogeneous and inhomogeneous problems

$$\begin{cases} \bar{x}_i = \bar{x}_{i-1} + a_i \bar{y}_i, \\ \bar{y}_i = \bar{y}_{i-1} + d_i \bar{x}_{i-1}, \\ \bar{x}_0 = 0, \quad \bar{y}_1 = 1, \end{cases} \begin{cases} \tilde{x}_i = \tilde{x}_{i-1} + a_i \tilde{y}_i + b_i, \\ \tilde{y}_i = \tilde{y}_{i-1} + d_i \tilde{x}_{i-1} + e_i, \quad i = 1,...,N \\ \tilde{x}_0 = 0, \quad \tilde{y}_1 = 0. \end{cases} \quad (12)$$

The unknowns $\tilde{x}_N$ and $\bar{x}_N$ can be found straightforwardly from the recurrence relations (12).

Finally, the last equation, $s_N'''(0, \mathbf{X}, \mathbf{Y}, \boldsymbol{\alpha}) = Z_N''^0$, allows to find α_3. This equation can be rewritten in a system of linear difference equations like (10) with some of the coefficients being different:

$$x_i \equiv s_i''', \; y_i \equiv E_i^0 \delta_i''', \; x_N^0 \equiv Z_N''^0$$

$$b_i = -6 t_{56i} \delta' \delta'' - 6 u_{56i} (\delta_i')^3,$$

$$e_i = k^3 Z_{i-1}^3 Y_i - 3k^2 Z_{i-1}^2 Z_{i-1}' X_i.$$

Hence, we have found a unique solution of the original problem (1) for any number of stages N. The explicit form of the solution for two and three stage bunch compression problem can be found in [1], [6].

Analytical Estimation of RF Tolerances

The final bunch length and the peak current are sensitive to the energy chirp and thus to the precise values of the RF parameters. Let us calculate a change of the compression due to a change of the RF parameters.

To simplify the notation we define

$$X_1 = E_0^0 + X_{1,1} + X_{1,3}, \; Y_1 = -\frac{\xi_1}{k} + Y_{1,1} + 3Y_{1,3},$$

where $\xi_1 = \partial_s E_0(0)$ is an initial energy chirp. Additionally we introduce RF parameter vectors

$$\mathbf{v}_i \equiv (X_i, Y_i)^T, \; \mathbf{v}_i^0 \equiv (X_i^0, Y_i^0)^T, \; \Delta\mathbf{v}_i \equiv (\Delta X_i, \Delta Y_i)^T,$$

$$X_i = X_i^0 + \Delta X_i, \; Y_i = Y_i^0 + \Delta Y_i,$$

where symbol "0" stays for the RF parameters as obtained in the previous section from the analytical solution.

In order to obtain a stable bunch compression and to estimate the acceptable change in the RF parameters we require that the relative change of compression $C_i \equiv Z_i^{-1}$ at $s = 0$ is smaller than Θ

$$\left| \left(C_i(\mathbf{v}_j) - C_i(\mathbf{v}_j^0) \right) / C_i(\mathbf{v}_j^0) \right| \le \Theta.$$

Neglecting the second order terms the last inequality can be rewritten in the form $\left| \Delta\mathbf{v}_j \cdot \nabla_{\mathbf{v}_j} C_i(\mathbf{v}_j) \right| \le C_i(\mathbf{v}_j^0)\Theta$,

where term $\nabla_{\mathbf{v}_j} C_i = \left(\partial_{X_j} C_i, \partial_{Y_j} C_i \right)^T$ means the gradient of the compression in two dimensional space (X_i, Y_i). Applying the Cauchy–Bunyakovsky inequality we obtain the admissible relative change in RF parameters (X_i, Y_i)

$$\left| \Delta\mathbf{v}_j \right| / \left| \mathbf{v}_j^0 \right| \le \left(Z_i^0 \Theta \right) / \left(V_j \left| \nabla_{\mathbf{v}_j} Z_i \right| \right), \; \Delta\mathbf{v}_i \equiv (\Delta X_i, \Delta Y_i)^T.$$

Hence, in order to estimate the RF tolerances we need to estimate the partial derivatives relative to the RF parameters (see [6] for the details).

It is shown in [6] that the lengths of the gradient vectors of the compression immediately after compressor BC_2 are given by relations

$$\left| \nabla_{\mathbf{v}_{1,1}} Z_2 \right| = k \frac{|r_{561} r_{562}|}{E_1 E_2} \sqrt{A_2^2 + B_2^2}, \quad (13)$$

$$A_2 = \left(E_2 r_{562}^{-1} + E_1 r_{561}^{-1} + k Y_2 \right),$$

$$B_2 = \left(\begin{aligned} & k X_2 Z_1 + 2 \frac{t_{561}}{r_{561}} \left(\frac{E_2}{r_{562}} + k Y_2 \right) \frac{\delta_1'}{k} + \\ & + 2 \frac{t_{562}}{r_{562}} \left(\frac{E_1}{r_{561}} + k Y_2 \right) \frac{\delta_2'}{k} \end{aligned} \right),$$

$$\left| \nabla_{\mathbf{v}_{1,3}} Z_2 \right| = k \frac{|r_{561} r_{562}|}{E_1 E_2} \sqrt{9 A_2^2 + B_2^2}.$$

If we neglect the non-linear compression terms and use Eqs. (7)-(9) then we can write the simple estimations

$$\left| \nabla_{\mathbf{v}_{1,1}} Z_2 \right| \approx \frac{k}{E_1 E_2} \sqrt{ \begin{aligned} & \frac{(E_1 r_{562} + E_2 r_{561} Z_2)^2}{Z_1^2} + \\ & + r_{561}^2 r_{562}^2 k^2 [E_2 - E_1]^2 Z_1^2 \end{aligned} }, \quad (14)$$

$$\left| \nabla_{\mathbf{v}_{1,3}} Z_2 \right| \approx \frac{k}{E_1 E_2} \sqrt{ \begin{aligned} & 9 \frac{(E_1 r_{562} + E_2 r_{561} Z_2)^2}{Z_1^2} + \\ & + r_{561}^2 r_{562}^2 k^2 [E_2 - E_1]^2 Z_1^2 \end{aligned} }.$$

Finally, let us consider a question about the best compression scenario from the point of view of the best possible tolerance in the booster $M_{1,1}$. We consider the two stage bunch compression scheme and use the approximate equation (14) to find the best value of Z_1 for the fixed value of Z_2. From the condition $\partial_{Z_1} \left| \nabla_{\mathbf{v}_{1,1}} Z_2 \right| = 0$ it is easy to find that the optimal value of the compression in the first bunch compression reads

$$Z_1 = \sqrt{ \frac{-r_{562} E_1 - r_{561} E_2 Z_2}{k r_{561} r_{562} (E_2 - E_1)} }. \quad (15)$$

In a companion paper [1] we apply the developed formalism to study the bunch compression schemes at FLASH and the European XFEL.

REFERENCES

[1] I. Zagorodnov, Ultra-short low charge operation at FLASH and the European XFEL, these Proceedings

[2] Ackerman W. et al., Nature Photonics 1, 336 (2007).

[3] M. Altarelli et al (Eds), DESY Report No. DESY 2006-097, 2006.

[4] K. Floettman, T. Limberg, Ph. Piot, DESY Report No. TESLA-FEL 2001-06, 2001.

[5] K. Togawa, T. Hara, H. Tanaka, Phys. Rev. ST Accel. Beams 12, 080706 (2009).

[6] I. Zagorodnov, M. Dohlus, DESY 10-102, 2010.