MODELING OF THE QUIET START ALGORITHM IN THE FRAMEWORK OF THE CORRELATION FUNCTION THEORY

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Abstract

To suppress initial beam current fluctuations at the fundamental harmonic the macroparticle-based FEL simulation codes use the quiet start algorithm. This algorithm should be valid at linear stage but there is no simple method to check whether it gives correct results at saturation. The regular approach to the start-up from noise problem should be based on the correlation function equation. In this paper we show that the quiet start algorithm can be naturally described in the framework of the correlation function theory. For this purpose one just needs to assume nonzero correlations in the initial particle distribution. This approach gives the possibility to compare simulation results for the system with reduced number of particles and artificially suppressed initial fluctuations with the case of real system with large number of particles.

INTRODUCTION

To model beam dynamics most of the FEL simulation codes use macroparticles. The charge of one macroparticle is much larger then electron charge which leads to significant increase of charge density fluctuations induced by shot noise. To suppress appearance of these fluctuations one uses specially constructed initial particle distribution in which particles are assembled in small groups called beamlets. Particles in one beamlet have the same energy and transversal coordinates. The distance in longitudinal direction between two adjacent particles is close to half of the radiation wavelength so, that radiation of individual particles in one beamlet is almost cancelled. This approach is known as quiet start algorithm (see, e. g., [1]).

Validity of the quiet start algorithm can be justified for the linear stage of SASE FEL operation but it is not so evident at saturation. The question of validity of this approach becomes especially significant when one tries to predict the saturation spectrum.

The suppression of initial fluctuations can be described within the framework of the correlation function theory [2]. In this theory the quiet start distribution can be considered as initial distribution with non-zero correlation function. The advantage of this approach is that it gives a regular way to check whether the evolution of the system with artificially suppressed fluctuations is equivalent to the evolution of the real system with large number of particles at saturation stage.

It also worth noting that initial correlations which suppress shot noise fluctuations can be created experimentally in real beam [3-5]. This phenomenon can be also described by correlation function theory.

In this paper we show how the non-zero initial correlation function can compensate the shot noise term in correlation function equation for the case of quiet start loading as well as for the case of real beam.

PARTICLE MOTION EQUATION

In this paper we consider the model of coasting beam and the half-infinite undulator which starts at $z=0$. The beam particle dynamics can be described by the following system of motion equations [2]:

$$\begin{aligned}
&\frac{dz^{(k)}}{dt}=1-\frac{1}{2\gamma_{//}^2}+\frac{\Delta^{(k)}}{\gamma_{//}^2}-\Delta\beta\left(z^{(k)},X^{(k)}\right)\\
&\frac{d\Delta^{(k)}}{dt}=\sum_{l\neq i}\Phi\left[z^{(k)},X^{(k)},z^{(l)}\left(t'^{(l)}\right),X^{(l)}\right]\\
&t-z^{(k)}=t'^{(l)}-z^{(l)}\left(t'^{(l)}\right)
\end{aligned}\tag{1}$$

where $z^{(k)}$ are particle longitudinal coordinates, $\Delta^{(k)}$ are relative energy deviations and $X^{(k)}$ are particle initial coordinates in 4-D transverse phase space. It is not a system of ordinary differential equations (ODE) as its right hand side (r.h.s.) is taken in retarded moment of time. But it can be reduced to ODE by the following choice of independent variable $\theta=2\gamma_{//}^2(t-z)$. For simplicity we shell restrict our consideration to 1-D case.

The resulting system of motion equations has the following form:

$$\begin{aligned}
&\frac{dz^{(k)}}{d\theta}=1+2\Delta^{(k)}\\
&\frac{d\Delta^{(k)}}{d\theta}=\sum_{l\neq k}\Phi\left(z^{(k)},z^{(l)}\right)
\end{aligned}\tag{2}$$

Further we shell use $\lambda_w/2\pi=1/k_w$ as the unit length for the longitudinal coordinates, where λ_w is the undulator period. The explicit expression for the interaction force $\Phi(z_1,z_2)$ can be obtained from the solution of radiation field equation. But for our purpose we just need to assume that this force obeys the following conditions:

1. $\Phi(z_1,z_2)=0$ for $z_1<0$ or $z_2<0$ or $z_1<z_2$. It means that particles do not interact outside undulator and front particles do not act on back ones.
2. $\Phi(z_1,z_2)=\Phi(z_1-z_2)$ inside undulator.

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3. $\Phi(\pi)=0$ - particles separated by half of radiation wavelength do not interact with each other.

4. $\Phi(z+\pi)=-\Phi(z)$. As the result of this condition the force acting from two particles separated by half of radiation wavelength on the particle in front of them is cancelled.

5. $\Phi(z)$ has narrow spectral bandwidth with central frequency $\omega=1$ so, that $\Phi''(z)\approx-\Phi(z)$.

Solution of Eq. (2) totally describes the FEL operation. To obtain it one just needs to specify particle initial coordinates. But in practice this solution cannot be obtained because of the large number of particles. In simulations this number is reduced by using of macroparticles. It should be noted that majority of the FEL simulation codes use the field amplitude as additional variable. So, the particles do not interact with each other directly. Using of macroparticles leads to increasing of the shot noise which results in increasing of spontaneous emission. In our model the field variable is excluded but this difference is not principal.

THE QUIET START INITIAL PARTICLE DISTRIBUTION

To describe the quiet start algorithm within the correlation function approach we need to find initial distribution function which corresponds to quiet start loading. Let us consider the beam with length L and number of particles N which enters the undulator. The coasting beam limit will be obtained further by increasing the beam length $L\to\infty$ and particle number $N=2m\to\infty$ so, that $N/L\to N_L$, where N_L .is the number of particles per the unit length. We assume that particles are grouped in m pairs. The particles in one pair have the same energy. The difference of their spatial coordinates is equal to $\pi+\delta z$ where δz is small deviation which is randomly distributed around zero value with density of probability $\chi(z)$. We assume, that $\chi(z)=\chi(-z)$ and $\int z^2\chi(z)dz=\sigma_\chi^2<<1$. The centres of these pairs are uniformly distributed along axes Z and their energies have distribution $F_0(\Delta)$.

The N - particle distribution function for such beam can be written in the following form:

$$f_N=\frac{1}{N!}\sum_{(i_1,\cdots,i_N)}\tilde{f}(i_1,i_2)\cdot\ldots\cdot\tilde{f}(i_{2m-1},i_{2m}) \tag{3}$$

where $\tilde{f}(i_1,i_2)=LF_0(\Delta_{i_1})\delta(\Delta_{i_1}-\Delta_{i_2})\chi(z_{i_1}-z_{i_2}-\pi)$

The sum is taken over all transpositions of particle numbers. This distribution function obeys the following normalizing condition:

$$\frac{1}{L^N}\int f_N(z_1,\Delta_1,\ldots,z_N,\Delta_N)dz_1d\Delta_1\ldots dz_Nd\Delta_N=1$$

Integrating of (3) over $N-2$ particle coordinates and taking the limit $L\to\infty$ one obtains the following expression for the two-particle distribution function:

$$\begin{aligned}f_2(z_1,\Delta_1;z_2,\Delta_2)&=F_0(\Delta_1)F_0(\Delta_2)+\\&+\frac{1}{2N_L}F_0(\Delta_1)\delta(\Delta_1-\Delta_2)(\chi(z_1-z_2-\pi)+\chi(z_2-z_1-\pi))\end{aligned} \tag{4}$$

The corresponding initial correlation function has the following form:

$$\begin{aligned}G_0(z_1,\Delta_1;z_2,\Delta_2)&=f_2(z_1,\Delta_1;z_2,\Delta_2)-F_0(\Delta_1)F_0(\Delta_2)=\\&=\frac{1}{2N_L}F_0(\Delta_1)\delta(\Delta_1-\Delta_2)(\chi(z_1-z_2-\pi)+\chi(z_2-z_1-\pi))\end{aligned} \tag{5}$$

CORRELATION FUNCTION EQUATION

Expression (5) is derived for some given moment of time θ. But one can easily check that for $z_1<0$ or $z_2<0$ it obeys the stationary correlation function equation:

$$\begin{aligned}&\left[(1+\Delta_1+\Delta_2)\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}\right)+(\Delta_1-\Delta_2)\left(\frac{\partial}{\partial z_1}-\frac{\partial}{\partial z_2}\right)\right]G(z_1,\Delta_1;z_2,\Delta_2)+\\&+N_L\frac{\partial}{\partial\Delta_1}F(z_1,\Delta_1)\int_0^{z_1}\int_{-\infty}^{\infty}\Phi(z_1-z_3)G(z_2,\Delta_2;z_3,\Delta_3)d\Delta_3dz_3\quad+\\&+N_L\frac{\partial}{\partial\Delta_2}F(z_2,\Delta_2)\int_0^{z_2}\int_{-\infty}^{\infty}\Phi(z_2-z_3)G(z_1,\Delta_1;z_3,\Delta_3)d\Delta_3dz_3\quad=\\&=-\left(\Phi(z_1-z_2)\frac{\partial}{\partial\Delta_1}+\Phi(z_2-z_1)\frac{\partial}{\partial\Delta_2}\right)F(z_1,\Delta_1)F(z_2,\Delta_2)\end{aligned} \tag{6}$$

So, we can try to find the solution of (6) in the following form:

$$G(z_1,\Delta_1;z_2,\Delta_2)=G_0(z_1,\Delta_1;z_2,\Delta_2)+g(z_1,\Delta_1;z_2,\Delta_2) \tag{7}$$

where $g(z_1,\Delta_1;z_2,\Delta_2)=0$ for $z_1<0$ or $z_2<0$.

Substituting (6) into (7) and taking into account that

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$$\int \Phi(z-t)\chi(t)dt \approx \int \left(\Phi(z) - t\Phi'(z) + \frac{1}{2}t^2\Phi''(z) \right)\chi(t)dt =$$

$$= \Phi(z) + \frac{\sigma_\chi^2}{2}\Phi''(z) \approx \Phi(z)\left(1 - \frac{\sigma_\chi^2}{2}\right)$$

we obtain the equation for $g(z_1,\Delta_1;z_2,\Delta_2)$:

$$\left[(1+\Delta_1+\Delta_2)\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}\right)+(\Delta_1-\Delta_2)\left(\frac{\partial}{\partial z_1}-\frac{\partial}{\partial z_2}\right)\right]g(z_1,\Delta_1;z_2,\Delta_2)+$$
$$+N_L\frac{\partial}{\partial \Delta_1}F(z_1,\Delta_1)\int_0^{z_1}\int_{-\infty}^{\infty}\Phi(z_1-z_3)g(z_2,\Delta_2;z_3,\Delta_3)d\Delta_3 dz_3 \quad +$$
$$+N_L\frac{\partial}{\partial \Delta_2}F(z_2,\Delta_2)\int_0^{z_2}\int_{-\infty}^{\infty}\Phi(z_2-z_3)g(z_1,\Delta_1;z_3,\Delta_3)d\Delta_3 dz_3 \quad =$$
$$= \quad -\frac{\sigma_\chi^2}{2}\left(\Phi(z_1-z_2)\frac{\partial}{\partial \Delta_1}+\Phi(z_2-z_1)\frac{\partial}{\partial \Delta_2}\right)F_0(\Delta_1)F_0(\Delta_2)-$$
$$-\Phi(z_1-z_2)\frac{\partial}{\partial \Delta_1}F(z_1,\Delta_1)(F(z_2,\Delta_2)-F_0(\Delta_2))-$$
$$-\Phi(z_2-z_1)\frac{\partial}{\partial \Delta_2}F(z_2,\Delta_2)(F(z_1,\Delta_1)-F_0(\Delta_1)) \tag{8}$$

The last two terms in the r.h.s. of (8) vanish at linear stage as at this stage $F(z,\Delta)=F_0(\Delta)$ So, we can see that at linear stage eq. (8) has the same form as eq. (6) but now the noise level which is determined by the first term in r.h.s. of eq. (8) depends on σ_χ . If we take $\sigma_\chi = \sqrt{\dfrac{2N_L}{N_L^{\exp}}}$ we obtain the same correlation function equation as for the beam which contains $N_L^{\exp}$ uncorrelated particles per unit length. It means that at linear stage behaviour of artificially constructed beam with N_L particles per unit length and quiet start distribution is similar to the behaviour of real beam with $N_L^{\exp}$ particles per unit length and uniform distribution.

Compensation of the shot noise is destroyed at saturation when $F(z,\Delta)$ changes significantly. The validity of quite start algorithm at this stage can be checked by numerical solution of eq. (8).

SUPPRESSION OF THE SHOT NOISE IN REAL BEAM

We considered how the correlation function theory describes suppression of the shot noise in electron beam with artificially constructed particle distribution. But this theory also describes the noise suppression mechanism in real beam with some initial correlations. To make this suppression one needs to create the particle distribution with initial correlation function $G_0(z_1,\Delta_1;z_2,\Delta_2)$ which obeys the following equations:

$$\left[(1+\Delta_1+\Delta_2)\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}\right)+(\Delta_1-\Delta_2)\left(\frac{\partial}{\partial z_1}-\frac{\partial}{\partial z_2}\right)\right]G_0=0$$

$$\int_0^{z_1}\int_{-\infty}^{\infty}\Phi(z_1-z_3)G_0(z_2,\Delta_2;z_3,\Delta_3)d\Delta_3 dz_3 \approx -\frac{F_0(\Delta_2)}{N_L}\Phi(z_1-z_2) \tag{9}$$

The first equation guarantees that correlations are not destroyed in undulator and the last one ensures compensation of the noise term in the r.h.s. of (6). Approximate equality sign in the last equation is used because in general case this equation may not have exact solution.

Let us consider an example of coasting beam with small energy spread. In this case $F_0(\Delta)\approx\delta(\Delta)$ and G_0 can be taken in the following form:

$$G_0(z_1,\Delta_1;z_2,\Delta_2)=\delta(\Delta_1)\delta(\Delta_2)\mu(z_1-z_2) \tag{10}$$

where $\mu(z_1-z_2)$ should be determined from the equation:

$$\int_0^{z_1}\Phi(z_1-z_3)\mu(z_2-z_3)dz_3 \approx -\frac{1}{N_L}\Phi(z_1-z_2) \tag{11}$$

If we take $\Phi(z_1-z_2)\sim\cos(z_1-z_2)$ then solution of eq. (9) has the following form:

$$\mu(z_1-z_2)=-\frac{2}{N_L}\cos(z_1-z_2)\Pi(z_1-z_2)$$

where $\Pi(z)=1/\pi$ if $z\in(-\pi/2,\pi/2)$ and $\Pi(z)=0$ otherwise. This solution is valid for $z_2\in(\pi/2,z_1-\pi/2)$.

Particle distribution with required initial correlation function can be obtained from the uniform distribution by "measuring" particle positions and making small shifts in longitudinal direction as it is suggested in [4].

CONCLUSION

In this paper we considered the shot noise suppression using the correlation function equation. It is interesting for both computer simulation and noise reduction in real beams. For linear problems the results may be obtained by other approaches. The reason of it is that for the linear transformations one can use the Green function of a problem and averaging over the ensemble of initial

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conditions. But, in nonlinear cases, like the FEL saturation, the correlation function technique may be the only appropriate one. Moreover, the correlation function approach gives us the different insight for statistical phenomena.

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