SMALL SIGNAL GAIN FOR TWO STREAM FEL

A. Raghavi[1] N. Mahdizadeh[2]
[1]Payame Noor University, 19395-4697, Tehran, Iran
[2]Department of Physics, Sabzevar Branch, Islamic Azad University, Sabzevar, Iran

Abstract

The small signal gain for a Two-Stream Free Electron Laser (TSFEL) is considered using a classical approach in a relativistic moving frame. The problem is considered non-relativistic in this frame and SSG is derived for both helical and planar wigglers. It is shown that in the ponderomotive frame the results are the same as the case of the relativistic consideration.

INTRODUCTION

Electrons in a FEL structure propagate with a relativistic bulk velocity near to the resonant velocity of FEL, namely, the phase velocity of the ponderomotive wave. In the context of a frame moving with the ponderomotive phase velocity, both longitudinal and transverse fluctuations of electron velocity are non-relativistic. Using a moving frame (MF) is a well-known method in the theoretical studies of FEL [1-4]. According to the non-relativistic nature of electron's fluctuations in MF, it looks relevant to deal the FEL interaction in this moving frame quite classically. Here we have employed a non-relativistic Hamiltonian approach in the ponderomotive frame to drive SSG for both helical and planar wigglers of a TSFEL. Transverse momentum is included in the Hamiltonian of the particle by means of its steady state value in the presence of the wiggler field.

SINGLE PARTICLE HAMILTONIAN IN THE PONDEROMOTIVE MF

The FEL structure we are going to consider here is composed of a helical wiggler magnetic field of amplitude B_w and wavenumber $k_{lw} = 2\pi/\lambda_{lw}$ and two relativistic electron beams of density $n_{ei}(i=1,2)$ propagating along the symmetry axis of the wiggler field. An electromagnetic(EM) wave will be amplified in this structure under a certain resonance condition. In the laboratory frame (LF), the vector potential of helical wiggler field can be described as

$$\vec{A}_{lw} = -A_w\left(\cos k_{lw}z_l\hat{e}_x + \sin k_{lw}z_l\hat{e}_y\right), \qquad (1)$$

where $A_w = B_w/k_{lw}$ is the amplitude of the vector potential. The amplified EM radiation then will be a circularly polarized wave with the following vector potential,

$$\delta\vec{A}_{ls} = -\delta A_{ls}(z_l,t_l)\begin{pmatrix}\cos[k_{ls}z_l - \omega_{ls}t_l + \delta(z_l,t_l)]\hat{e}_x - \\ \sin[k_{ls}z_l - \omega_{ls}t_l + \delta(z_l,t_l)]\hat{e}_y\end{pmatrix}. \qquad (2)$$

Where k_{ls} and $\omega_{ls} = ck_{ls}$ are its wavenumber and frequency, respectively. Here we choose to work in the Compton regime where the scalar potential of the wave can be neglected (i.e. $\delta\varphi_l = 0$). In the context of a moving frame (MF) moving with the velocity of ponderomotive wave, that is $u_p = \beta_p c = k_{ls}/(k_{lw}+k_{ls})$, the FEL parameters are

$$\vec{A}_w = -A_w\left[\cos(k_w z + \omega_w t)\hat{e}_x + \sin(k_w z + \omega_w t)\hat{e}_y\right], \qquad (3)$$

$$\delta\vec{A}_s = -\delta A_s(z,t)\begin{pmatrix}\cos[k_s z - \omega_s t + \delta(z,t)]\hat{e}_x - \\ \sin[k_s z - \omega_s t + \delta(z,t)]\hat{e}_y\end{pmatrix}, \qquad (4)$$

where $z = \gamma_p(z_l - u_p t_l)$, $t = \gamma_p(t_l - u_p z_l/c^2)$, $k_w = \gamma_p k_{lw}$ and $k_s = \gamma_p(1-\beta_p)k_{ls}$ are the FEL parameters in MF, $\gamma_p = (1-\beta_p^2)^{-1/2}$ is the Lorentz factor of MF and $\omega_w = u_p k_w$. The Lorentz factor of the particles in MF is

$$\gamma = \gamma_p(\gamma_l - u_p p_{lz}/m_e c^2). \qquad (5)$$

Since we are working in the ponderomotive MF, it will be acceptable to suppose the electrons that are moving with non-relativistic velocities, such that we can write the non-relativistic Hamiltonian for a single particle of charge e and rest mass m_e as

$$H = \frac{1}{2m_e}\left(\sum\vec{p}_i - \frac{e}{c}\vec{A}\right)^2. \qquad (6)$$

Here $\vec{p}$ is classical conjugate momentum of particles and $\vec{A} = \vec{A}_w + \delta\vec{A}_s$ is the total vector potential. According to (1) and (2), $\vec{A}$ is a fully transverse vector therefore, we can rewrite (6) as

$$H = \frac{1}{2m_e}\left\{\sum p_{iz}^2 + \left(\sum\vec{p}_{i\perp} - \frac{e}{c}\vec{A}\right)^2\right\}, \qquad (7)$$

Since Hamiltonian of the system is independent of transverse coordinates x and y, then transverse component of conjugate momentum, namely $\sum\vec{p}_\perp = \sum\vec{P}_\perp + (e/c)\vec{A}$, is a constant of motion, where $\vec{P}_\perp$ is the transverse mechanical momentum. According to the well known equation $\sum\vec{P}_\perp = -(e/c)\vec{A}$ which holds for the case of $\delta\varphi = 0$ [3] the value of this constant is zero and the Hamiltonian should be written as

$$H = \frac{1}{2m_e}\left(\sum p_z^2 + \frac{e^2}{c^2}A^2\right). \qquad (8)$$

Substituting (3) and (4) into (8) results

$$H = \frac{1}{2m_e}\left\{\sum p_z^2 + \frac{e^2}{c^2}\left[A_w^2 + 2A_w\delta A_s\cos(kz + \Delta\omega t + \delta)\right]\right\}, \qquad (9)$$

ISBN 978-3-95450-117-5

where $k = k_w + k_s$ and $\Delta\omega = \omega_w - \omega_s$. Also, according to the fact that $A_w^2 >> \delta A_s^2$ we have dropped δA_s^2 from RHS of (9). Referring to the ponderomotive MF, where $\Delta\omega = 0$ and $k = k_{ls}/\beta\gamma_p$, we end with the following single particle Hamiltonian

$$H = \frac{1}{2m_e}\left\{\sum p_z^2 + \frac{e^2}{c^2}\left[A_w^2 + 2A_w\delta A_s \cos(kz+\delta)\right]\right\}. \quad (10)$$

Supposing $\delta A_s(z,t)$ and $\delta(z,t)$ as slow-varying functions of time (corresponding to small signal gain regime approximation), the canonical equations of motion derive from (10) as

$$\dot{p}_{zi} = -\frac{\partial H}{\partial z_i} = -\frac{e^2 k A_w \delta A_s}{m_e c^2}\sin(kz_i + \delta),$$

$$\dot{z}_i = \frac{\partial H}{\partial p_{zi}} = \frac{p_{zi}}{m_e}. \quad (11)$$

SMALL SIGNAL GAIN

Small signal gain of the FEL defines as the average variation in the energy of wave to its initial energy. Since energy exchange occurs only between the e-beam and the wave, the variation of the wave energy is equal to the total change in the e-beam energy. In the LF, the change in the energy of a single electron is $\Delta\varepsilon = -\Delta\gamma_l m_e c^2$. If A is the cross-section of the interaction region and electrons move with an axial velocity v_{lz} during the interaction time T_l, then the number of electrons that have contribution in this interaction is $n_e A v_{lz} T_l$. Therefore, total change of the e-beam energy is

$$\Delta E = -\sum(\Delta\gamma_{li} m_e c^2)(n_{ei} A v_{lzi} T_{li}). \quad (12)$$

Total energy of the wave obtains from its energy density which is

$$u_E = \frac{1}{8\pi}(E_s^2 + B_s^2) = \frac{k_{ls}^2}{4\pi}\delta A_s^2 \quad (13)$$

The volume that wave has occupied during the interaction time T_l is $V = AcT_l$. This, together with (13) gives the following value for the total energy of wave

$$E_t = \frac{AcT_l k_{ls}^2}{4\pi}\delta A_s^2. \quad (14)$$

The radiation energy gain per pass therefore obeys

$$G = \sum \frac{4\pi n_{ei}\beta_{lzi} m_e c^2}{k_{ls}^2 \delta A_s^2}\Delta\gamma_{li}. \quad (15)$$

Here

$$\Delta\gamma_{li} = \int_0^{T_{li}} \left\langle \frac{d\gamma_{li}}{dt} \right\rangle_{\psi_{0i}} dt \quad (16)$$

where $\langle\cdots\rangle_{\psi_0} = \frac{1}{2\pi}\int_0^{2\pi}(\cdots)d\psi_0$ is average over the all initial phases of electrons. From (5) one get to the following equation for $\Delta\gamma$ in LF and MF

$$\Delta\gamma_{li} = \gamma_p(\Delta\gamma_i + u\Delta p_{zi}/m_e c^2). \quad (17)$$

Because of the non-relativistic motion of electrons in MF we are permitted to let $\Delta\gamma_i = 0$. Subsequently, to specify $\Delta\gamma_{li}$ in LF it will be enough to evaluate Δp_{zi} in MF. The value of Δp_z can be evaluated from the equations of motion are given in equations. (11) and (12). In order to solve this set of linear differential equations, we use an iteration method similar to what is used in [4]. To this end we introduce a set of new variables as $\psi_i = kz_i$, $q_i = p_{iz}$, $x = (k/m_e)t$ and $\alpha = e^2 A_w \delta A_s / c^2$.

Using the new variables equations (11) and (12) becomes

$$\frac{dq_i}{dx} = -\alpha\sin(\psi_i + \delta), \quad (18)$$

$$\frac{d\psi_i}{dx} = q_i. \quad (19)$$

We now expand unknown functions via α, which is a small parameter, and get

$$\psi_i = \sum_{n=0}^{\infty}\psi_i^{(n)}\alpha^n, \quad (20)$$

$$q_i = \sum_{n=0}^{\infty} q_i^{(n)}\alpha^n, \quad (21)$$

$$\alpha\sin(\psi_i+\delta) = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial\alpha^n}[\alpha\sin(\psi_i+\delta)]_{\alpha=0}\alpha^n. \quad (22)$$

Instituting these quantities into (18) and (19) we find the perturbation equations as

$$\frac{dq_i^{(n)}}{dx} = -\frac{1}{n!}\frac{\partial^n}{\partial\alpha^n}[\alpha\sin(\psi_i+\delta)]_{\alpha=0}, \quad (23)$$

$$\frac{d\psi_i^{(n)}}{dx} = q_i^{(n)}, \quad (24)$$

with the initial conditions as $\psi_i^{(n)}(0) = \psi_{0i}\delta_{n,0}$ and $q_i^{(n)}(0) = q_{0i}\delta_{n,0}$. Integrating these equations we find for the zero order answers

$$q_i^{(0)} = q_{0i} = const., \quad (25)$$

$$\psi_i^{(0)} = q_{0i}x + \psi_{0i} \quad (26)$$

Using these answers in (24) and (25) gets

$$\frac{dq_i^{(1)}}{dx} = -\sin(q_{0i}x + \psi_{0i} + \delta), \quad (27)$$

$$\frac{d\psi_i^{(1)}}{dx} = q_i^{(1)}. \quad (28)$$

Taking a phase average over (28) gives identically zero as it is simply a sine function of ψ_{0i}. Thus, to this order of approximation no gain occurs. In order to find the lowest order gain, we must carry out the next step in the approximation scheme. Using the answers of (28) and (29) in (24) and (25) we find the relation

ISBN 978-3-95450-117-5

$$\frac{dq_i^{(2)}}{dx} = -\frac{1}{q_{0i}^2}[\sin(q_{0i}x+\phi_{0i}) - \sin(\phi_{0i}) - q_{0i}x\cos(\phi_{0i})]\cos(q_{0i}x+\phi_{0i}) \tag{29}$$

When we average this over all values of ψ_{0i}, we get to

$$\left\langle \frac{dq_i^{(2)}}{dx} \right\rangle_{\psi_{0i}} = -\frac{1}{2q_{0i}^2}[\sin(q_{0i}x) - q_{0i}x\cos(q_{0i}x)] \tag{30}$$

Total variation for q during the total interaction time T, to this order of approximation, is

$$\Delta q_i = \alpha^2 \Delta q_i^{(2)} = \alpha^2 \int_0^X \left\langle \frac{dq_i^{(2)}}{dx} \right\rangle_{\psi_0} dx = \alpha^2 X^3 \frac{[\cos(q_{0i}X) - 1 + \frac{1}{2}q_{0i}X\sin(q_{0i}X)]}{q_{0i}^3 X^3} \tag{31}$$

Substituting this into (18) and then placing the result into (16) we get to the following expression for SSG

$$G_{Helical} = \frac{e^2 B_w^2 k_{ls}}{8\gamma_p^5 m_e^2 c^2 k_w^2} L_l^3 \sum_{i=1}^{2} \frac{\omega_{bi}^2}{v_{lzi}^4} F(q_{0i}X/2) \tag{32}$$

where $\omega_{bi}^2 = 4\pi n_{ei} e^2 / m_e$ is the plasma frequency of e-beam, $L_l = v_{lz} T_l$ is the interaction length and $F(q_0 X/2)$ is the well known spectral function

$$F(q_0 X/2) = \frac{\partial}{\partial(q_0 X/2)} \left[\frac{\sin(q_0 X/2)}{q_0 X/2} \right]^2. \tag{33}$$

For the case of a planar wiggler, which its magnetic field can be derived from the vector potential

$$\vec{A}_{lw} = -A_w \sin k_{lw} z_l \hat{e}_y \tag{34}$$

we follow the guide line for the helical wiggler to get to the following Hamiltonian for the single particles

$$H = \frac{1}{2m_e}\left\{ \sum p_{zi}^2 + \frac{e^2}{c^2}\left[A_w^2 \sin^2(k_w z + \omega_w t) + \delta A_s^2 \sin^2(k_s z - \omega_s t + \delta) - A_w \delta A_s \left(\cos(\Delta k z + \omega t - \delta) - \cos(kz + \Delta\omega t + \delta) \right) \right] \right\} \tag{35}$$

Where $\Delta k = k_w - k_s$ and $\omega = \omega_w + \omega_s$.

In the context of ponderomotive frame, only time dependency rise from ω_s and ω that both are high frequencies causing fast time variation of Hamiltonian. So, if we take a time average over total interaction time T, it gets a simplified form as

$$H = \frac{1}{2m_e}\left\{ \sum p_{zi}^2 + \frac{e^2}{2c^2}\left[A_w^2 + 2A_w \delta A_s(z,t)\cos(kz+\delta) \right] \right\} \tag{36}$$

This expression differs from its helical wiggler counterpart by a factor of 1/2 in the contribution of vector potentials. Accordingly, the corresponding SSG will differ by a factor of 1/2 that is

$$G_{Planar} = \frac{e^2 B_w^2 k_{ls}}{16\gamma_p^5 m_e^2 c^2 k_w^2} L_l^3 \sum_{i=1}^{2} \frac{\omega_{bi}^2}{v_{lzi}^4} F(q_{0i}X/2). \tag{37}$$

Now, we are going to find the resonant wavenumber corresponding to the maximum gain. For this, we refer to the spectral function (32), which has an absolute maximum at $q_{0i}X \approx -2.6$. To find this point in the LF we notice first that the ponderomotive phase ψ is a quantity which is independent of the working frame. Therefore, we can replace ψ with its LF value $\psi_l = k_l z_l - \omega_{ls} t_l$ in (19). Doing so, and applying the chain differentiating rule one find

$$qX = \frac{d\psi_l}{dx}X = \left(\frac{d\psi_l}{dz_l}\frac{dz_l}{dx} + \frac{d\psi_l}{dt_l}\frac{dt_l}{dx} \right)X = u_p[k_{lw} - k_{ls}(1-\beta_p)]T_l \tag{38}$$

Finally, the resonance wavenumber can be evaluated, under the assumption $u_p \approx v_{0z}$ and $1+\beta_{0z} \approx 2$ for a relativistic beam, as

$$k_{lsi} \approx 2\gamma_{0zi}^2 k_{lw}\left(1 + \frac{2.6}{k_{lw}L_l} \right) \tag{39}$$

which is the well-kn own FEL resonance wavenumber corresponding to the i'th electron beam. Although the resonance wavelengths of two beams are different, for the case of a small distinction between two electron beam energies, it is easy to show that

$$\frac{\Delta\lambda}{\lambda_w} \approx \frac{\Delta\gamma}{\gamma^3} << 1, \tag{40}$$

which is small enough for two wavelengths to be approximated as one wavelength.

CONCLUTIONS

We have calculated the small signal gain of a two-stream FEL for both helical and planar wigglers through a non-relativistic approach in the ponderomotive frame. The effect of transverse momentum of electrons is incorporated in the calculations by means of its steady state value in the presence of the wiggler field. Resulting expression for the gain shows that the total FEL gain in this situation is simply the sum of two conventional expressions for SSG for each electron beam, separately.

REFERENCES

[1] R.A. Bosch, 2003, Proc. of the 2003 Particle Accelerator Conf, 935.

[2] I. Boscolo, et al., 1985, J. Appl. Phys. **57(1),** 18

[3] R.C. Davidson, 2001, Physics of Nonneutral Plasmas, (World Scientific Publishing Co.).

[4] S.T. Stenholm and A. Bambini, 1981, IEEE J. quantum electronics, **QE-17**, 1363.

[5] H.P.Freund and J.M.Antonsen, 1996, Principles of Free Electron Lasers, (London Chapman & Hall).

[6] A. Raghavi, G. De Ninno and H. Mehdian, 2008, Nucl. Instrum. Meth A, **591**, 338.

[7] L. Friedland and J.L. Hirshfield, 1980, Phys. Rev. Lett., **44**, 1456.

ISBN 978-3-95450-117-5