REEVALUATION OF COHERENT ELECTRON COOLING GAIN FACTOR*

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Abstract

An important element in the concept of coherent electron cooling [1] is amplification of the electric field induced by a point charge in an electron beam passing through an FEL amplifier. We calculate this factor in 1D FEL theory and show that it is equal to the conventional FEL gain (for the field) multiplied by the relative bandwidth of the FEL amplifier, which is typically a small parameter of the order of 10^{-3}. The obtained amplification factor is more than two orders smaller than quoted in Ref. [1]. We also discuss the recent reply [2] of the authors of [1] to our comment [3] and show that critical remarks in the reply with regard to the comment are unjustified.

INTRODUCTION

In Ref. [1] the authors put forward a concept of coherent electron cooling of hadrons. At the core of the concept lies the following idea: a density perturbation induced by an hadron in a co-propagating electron beam is amplified by several orders of magnitude in a free electron laser (FEL). After the FEL the electron beam is merged again with the hadron one and the amplified electric field in the electron beam acts back on each hadron resulting, after many repetitions, in a cooling of the hadron beam. The efficiency of the process is critically determined by the amplification factor of the longitudinal electric field induced by the hadron in the electron beam. The authors associate this amplification with the FEL gain factor. In this note we show that it is actually considerably smaller than the (conventionally defined) FEL gain with the smallness parameter to be the relative bandwidth σ_ω/ω_0 of the FEL amplifier.

This paper is an expanded and detailed version of the comment [3] on the original publication [1].

AMPLIFICATION OF THE LONGITUDINAL FIELD INDUCED BY HADRON

In our analysis we use a standard one-dimensional linear FEL theory which gives a reasonably good approximation for typical parameters of modern FELs, (see, e.g., [4, 5]). For simplicity we assume a helical undulator with the undulator parameter K, the undulator period $\lambda_u = 2\pi/k_u$ and length l_u. An electron beam with a localized line density perturbation $\delta n_0(z)$ induced by an hadron (δn_0 has dimension of inverse length, z is the longitudinal coordinate inside the bunch in the direction of propagation) enters the FEL. Following [5] we use the dimensionless undulator length $\tau = k_u l_u$.

We expand $\delta n_0(z)$ into Fourier integral and use linear FEL theory to propagate each harmonic from the beginning to the end of the FEL assuming a high-gain FEL process. The density at the exit $\delta n_0(z,\tau)$ is Fourier transformed over z

$$\delta n_q(\tau) = \int_{-\infty}^{\infty} dz e^{-ik_0(1+q)z} \delta n_0(z,\tau), \tag{1}$$

where $k_0 = \omega_0/c = 2\gamma^2 k_u/(1+K^2)$ corresponds to the fundamental FEL frequency and q is the dimensionless detuning. In a linear approximation, assuming a cold beam, the FEL instability develops as $\delta n_q \propto e^{s\tau}$ with s satisfying the dispersion equation

$$s^2(s+iq) = i(2\rho)^3, \tag{2}$$

with ρ the standard FEL parameter defined by

$$(2\rho)^3 = \frac{2\lambda_u}{\gamma k_0 S}\frac{K^2}{1+K^2}\frac{I}{I_A}, \tag{3}$$

where γ is the beam Lorentz factor, S is the beam area, I is the beam current and $I_A = mc^3/e \approx 17$ kA is the Alfvén current. The three roots of (2), s_1, s_2 and s_3, for small detuning q, can be approximated [5] by

$$s_i \approx 2\rho\left[\mu_i - \frac{i}{3}\frac{q}{2\rho} - \frac{1}{9\mu_i}\left(\frac{q}{2\rho}\right)^2\right],\ i = 1,2,3, \tag{4}$$

with $\mu_1 = \frac{\sqrt{3}}{2} + \frac{i}{2}$, $\mu_2 = -\frac{\sqrt{3}}{2} + \frac{i}{2}$ and $\mu_3 = -i$. In what follows we assume a large gain, then the terms involving s_2 and s_3 can be neglected and only the fastest growing exponential term involving s_1 is kept. The Fourier transform $\delta n_q(\tau)$ at the exit of the FEL in this limit can be expressed through the initial value $\delta n_q(0)$ [5]

$$\delta n_q(\tau) = (s_1 + iq)H_q(\tau)\delta n_q(0), \tag{5}$$

where

$$H_q(\tau) = \frac{s_1 e^{s_1\tau}}{(s_1-s_2)(s_1-s_3)}. \tag{6}$$

Let us assume that $\delta n_0(z)$ corresponds to a localized perturbation at $z = 0$ that carries a charge Ze. If the width

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Δz of the perturbation is smaller than the reduced radiation wavelength $1/k_0$ it can be approximated by a delta function $\delta(z)$:

$$\delta n_0(z) = Z\delta(z), \tag{7}$$

and $\delta n_q(0) = Z$. Note that $E_0 = \pm 2\pi Ze/S$ is the initial electric field in the vicinity of the perturbation (7) (at distanced much smaller than $\sqrt{S}/\gamma$) in 1D model.

The density perturbation $\delta n(z, \tau)$ is given by the inverse Fourier transformation

$$\begin{aligned}\delta n(z,\tau) &= \frac{k_0}{2\pi}\int_{-\infty}^{\infty} dq e^{ik_0(1+q)z}\delta n_q(\tau) \qquad (8)\\ &= \frac{1}{2\pi}k_0 Z e^{ik_0 z}\int_{-\infty}^{\infty} dq e^{ik_0 zq}(s_1+iq)H_q(\tau).\end{aligned}$$

In the expression for $(s_1 + iq)H_q(\tau)$ we can neglect q in comparison with s everywhere, except in the exponent of $e^{s_1\tau}$, which with the help of (4) gives

$$\begin{aligned}s_1 H_q(\tau) = \frac{1}{3}\exp\Bigg(2\rho\tau\Bigg[&\frac{\sqrt{3}}{2}+\frac{i}{2}-\frac{i}{3}\frac{q}{2\rho}\\ &-\frac{1}{9}\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)\left(\frac{q}{2\rho}\right)^2\Bigg]\Bigg).\end{aligned} \tag{9}$$

We obtain

$$\begin{aligned}\delta n(z,\tau) = \frac{1}{6\pi}k_0 Z e^{ik_0 z+(\sqrt{3}+i)\rho\tau}&\int_{-\infty}^{\infty} dq e^{ik_0 zq}\\ \times\exp&\left(-\tau\left[\frac{i}{3}q+\frac{1}{9}\left(\sqrt{3}-i\right)\frac{q^2}{4\rho}\right]\right).\end{aligned} \tag{10}$$

The integral in (10) is easily computed

$$\begin{aligned}&\int_{-\infty}^{\infty} dq e^{ik_0 zq}\exp\left(-\tau\left[\frac{i}{3}q+\frac{1}{9}\left(\sqrt{3}-i\right)\frac{q^2}{4\rho}\right]\right)\\ &= \sqrt{\frac{\rho}{\tau}}\frac{6\sqrt{\pi}}{\sqrt{3^{1/2}-i}}\exp\left[-\frac{\rho(\tau-3k_0 z)^2}{\left(\sqrt{3}-i\right)\tau}\right].\end{aligned} \tag{11}$$

We see that for a given τ (the undulator length) the absolute value $|\delta n(z, \tau)|$ has a Gaussian distribution over z. The maximal value of $|\delta n(z, \tau)|$ is achieved at the point where the argument of the exponential function in (11) is equal to zero, $k_0 z = \tau/3$. Introducing the standard *power* gain length L_g,

$$L_g^{-1} = 2\sqrt{3}\rho k_u, \tag{12}$$

we replace $\rho\tau = l_u/2\sqrt{3}L_g$ and obtain

$$\max|\delta n(z,\tau)| = \frac{3^{1/4}}{\sqrt{\pi}}k_0 Z\rho\sqrt{\frac{L_g}{l_u}}e^{l_u/2L_g}. \tag{13}$$

The longitudinal electric field $\delta E_\parallel(z, \tau)$ generated by the density perturbation $\delta n(z, \tau)$ is found from the 1D Poisson equation. This equation is trivially solved if one remembers that $\delta n(z, \tau)$ has a fast oscillating factor $e^{ik_0 z}$ in it, hence

$$\begin{aligned}\max|\delta E_\parallel(z,\tau)| &= \frac{4\pi e}{k_0 S}\max|\delta n(z,\tau)|\\ &= \frac{4\pi Ze}{S}\frac{3^{1/4}}{\sqrt{\pi}}\rho\sqrt{\frac{L_g}{l_u}}e^{l_u/2L_g}.\end{aligned} \tag{14}$$

We can write the result (14) as the initial field E_0 multiplied by an amplification factor G, $\max|\delta E_\parallel(z, \tau)| = G|E_0|$, where

$$G = 2\frac{3^{1/4}}{\sqrt{\pi}}\rho\sqrt{\frac{L_g}{l_u}}e^{l_u/2L_g}. \tag{15}$$

This factor G can be related to the (amplitude) *FEL amplification factor* $G_{\rm FEL}$. The latter is usually defined as a ratio of the final (exit) amplitude of a sinusoidal density perturbation at the resonant frequency ($q = 0$) to its initial value; in our notation $G_{\rm FEL} = |\delta n_q(\tau)/\delta n_q(0)|_{q=0}$. Using (5) we find

$$G_{\rm FEL} = |s_1 H_q(\tau)|_{q=0} = \frac{1}{3}e^{l_u/2L_g}. \tag{16}$$

We see that the amplification factor G of the longitudinal field (15) is much smaller than the FEL amplification factor

$$G = 2\frac{3^{5/4}}{\sqrt{\pi}}\rho\sqrt{\frac{L_g}{l_u}}G_{\rm FEL}, \tag{17}$$

in contrast to the statement in [1] where it seems that G is identified with $G_{\rm FEL}$. Note that Eq. (17) can also be written as

$$G = \frac{\sigma_\omega}{\omega_0}G_{\rm FEL}, \tag{18}$$

where the relative FEL bandwidth is defined as[1]

$$\frac{\sigma_\omega}{\omega_0} = 2\frac{3^{5/4}}{\sqrt{\pi}}\rho\sqrt{\frac{L_g}{l_u}}. \tag{19}$$

Eq. (18) shows that the smallness of G in comparison with $G_{\rm FEL}$ is due to the narrow amplification line of the FEL. Given that the parameter ρ is a small quantity, of the order of 10^{-3}, the difference between $G_{\rm FEL}$ and G can be as large as two to three orders of magnitude.

NUMERICAL ESTIMATE

Note that, as discussed in [1], the maximally achievable FEL gain is limited by FEL saturation. In saturation the density modulation reaches the averaged density of the beam, or, equivalently, the bunching factor becomes of the order of one. The saturation length $l^{\rm sat}$ can be estimated from the linear FEL theory using an equation for the power of the FEL radiation which starts from the shot noise [5] (the SASE regime)

$$P(l) = \frac{1}{3\sqrt{\pi}}\rho^2\omega_0\gamma mc^2\sqrt{\frac{L_g}{l}}e^{l/L_g}. \tag{20}$$

[1] This definition differs by a numerical factor from [5], however the difference between the two definitions is less then 5%.

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It is known that in saturation the SASE FEL power is approximately equal to $\rho\gamma mc^2 I/e$. Equating this quantity to (20) we can express the ratio $l^{\rm sat}/L_g$ through other FEL parameters:

$$\sqrt{\frac{L_g}{l^{\rm sat}}}e^{l^{\rm sat}/L_g} = \frac{3}{2\sqrt{\pi}}\frac{\lambda_0}{\rho r_e}\frac{I}{I_A}, \qquad (21)$$

where $\lambda_0 = 2\pi/k_0$ is the FEL wavelength, and $r_e = e^2/mc^2$ is the classical electron radius.

We now use the parameters quoted in [1] for an hypothetical FEL for an LHC cooler: $\lambda_0 = 10$ nm, the undulator period $\lambda_u = 5$ cm, $I = 100$ A, $\gamma = 7.6\times 10^3$. From the relation between λ_0 and λ_u we find $K = 4.6$. We assume the electron beam emittance of $\epsilon_n = 3\ \mu$m (such a relatively large emittance is due to a large electron beam charge of several nC needed for CeC) and the beta function of $\beta = 10$ m in the undulator. Estimating the transverse area of the beam as $S = 2\pi\beta\epsilon_n/\gamma$ we find $S = 2.5\times 10^{-4}$ cm^2. From (4) we now find the parameter $\rho = 8.7\times 10^{-4}$ and the saturation length $l^{\rm sat}/L_g = 18.3$. Assuming $l_u = l^{\rm sat}$, Eq. (15) gives $G = 2.8$ which is more than two orders short of the value $G = 500$ assumed by the authors of [1].

SCALING OF THE GAIN FACTOR WITH BEAM PARAMETERS

In reply [2] to our comment [3] the authors claim that the result obtained in [3] (derived above as Eqs. (17) and (18)) is in error because it gives a wrong scaling of the gain factor with the beam and FEL parameters (and in particular with the FEL bandwidth). In this section we derive this scaling and show that it is in full agreement with [2], in contrast to the statement in that paper.

Solving (21) for $e^{l^{\rm sat}/2L_g}$ and substituting it into (15) we find the maximal gain $G_{\rm max}$ corresponding to the length of the undulator equal to the saturation length,

$$\begin{aligned} G_{\rm max} &= \sqrt{2}\left(\frac{3}{\pi}\right)^{3/4}\left(\frac{L_g}{l^{\rm sat}}\right)^{1/4}\left(\frac{\lambda_0\rho}{r_e}\frac{I}{I_A}\right)^{1/2} \\ &= \frac{3^{1/8}}{\sqrt{\pi}}\left(\frac{\sigma_\omega}{\omega_0}\right)^{1/2}\left(\frac{\lambda_0}{r_e}\frac{I}{I_A}\right)^{1/2} \\ &\approx 144\left(\frac{\sigma_\omega}{\omega_0}I[{\rm A}]\lambda_0[\mu{\rm m}]\right)^{1/2}, \end{aligned} \qquad (22)$$

where the last line gives the result in practical units. This equation coincides with Eq. (5) in [2] if one neglects the effects of the finite current of the proton beam, which is beyond the scope of this paper.

It follows from Eq. (22) that the amplification factor G is roughly proportional to the square root of the radiation wavelength λ_0. Hence choosing a larger wavelength can increase G (assuming that an undulator for such a wavelength is feasible). Some effects relevant for longer FEL wavelengths are considered in the next section.

USING LONG-WAVELENGTH FEL IN COHERENT ELECTRON COOLING

The 1D FEL theory used in the previous sections is valid if the beam cross section area S is larger than the product of the gain length and the inverse wave number of radiation, $S \gg L_g/k_0$. Using a large FEL wavelength can violate this inequality. In the opposite limit, $S \lesssim L_g/k_0$, one has to employ the 3D FEL model, in which discreet modes are amplified when the beam propagates through the undulator. While analysis in this case becomes more complicated (due to the lack of universality of the 1D model), the main effect of the narrowness of the FEL bandwidth remains valid, as well as our final result (18).

Increasing the wavelength λ_0 can also lead to suppression of the longitudinal electric field for a given amplitude of the density modulation δn. Instead of the 1D relation $\delta E_\parallel = (4\pi e/k_0 S)\delta n$ used in the previous section one has to solve a 2D Poisson equation for a given transverse density profile. Assuming a Gaussian profile and sinusoidal modulation along the beam, $\delta n = \delta n_0 \sin(k_0 z)(2\pi\sigma^2)^{-1}e^{-r^2/2\sigma^2}$, it is easy to find the electric field on the axis of the beam, $r = 0$,

$$\delta\mathcal{E}_\parallel(z) = \frac{2e\delta n_0}{k_0\sigma^2}\cos(k_0 z)\mathcal{J}\left(\frac{k_0\sigma}{\gamma}\right), \qquad (23)$$

where

$$\mathcal{J}(q) = \frac{1}{q^2}\int_0^\infty t dt[1 - tK_1(t)]e^{-t^2/2q^2}. \qquad (24)$$

The plot of function $\mathcal{J}(q)$ is shown in Fig. 1. In the limit

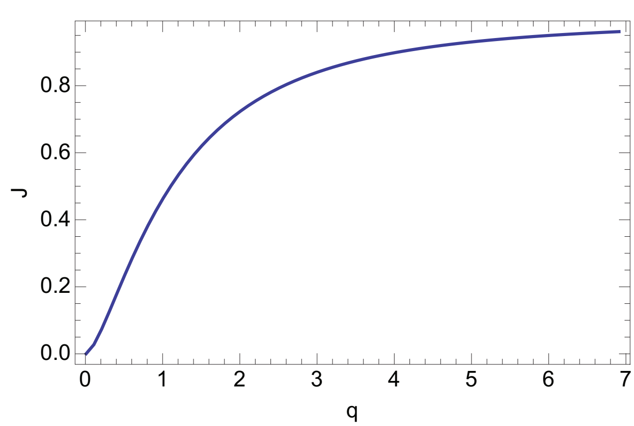

Figure 1: Function $\mathcal{J}(q)$.

$q \gg 1$ we have $\mathcal{J}(q) \to 1$ and one recovers the 1D result with S replaced by $2\pi\sigma^2$. In the opposite limit $q \ll 1$ the electric field on the axis diminishes.

Note that for parameters of the proof-of-principle installation [6] with the beam energy 21.8 MeV, the beam emittance 5 μm, the beta function 5.5 m, and the FEL wavelength 10 μm, the ratio $k_0\sigma/\gamma$ is approximately equal to

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12 and the suppression effect is negligible. However, for higher energy FELs, it may impose a certain restriction for the design of the cooler.

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