BROAD-BAND AMPLIFIER BASED ON TWO-STREAM INSTABILITY*

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Abstract

A broadband FEL amplifier is of great interests for short-pulse generation in FEL technology as well as for novel hadron beam cooling technique, such as CeC. We present our founding of a broadband amplification in 1D FEL dispersion relation based on electron beam with two energy peaks and a strong space charge forces. We connect its origin to the two-stream instability in electron plasma. Assuming a spatially uniform electron beam with double-peak κ-2 velocity distribution, we obtained a close form expression in the 3-D wave vector domain for the electron density variation induced by a point-like perturbation. The solution is then numerically inverse Fourier transformed to the configuration space.

INTRODUCTION

As observed from our previous studies [1], the 1D FEL dispersion relation has two growing modes for electron beam with double-peak energy distribution and sufficiently strong space charge. While one of the solutions has the typical narrow bandwidth of the FEL instability, the other solution has a much wider frequency range for amplification. After properly taking into account the frequency dependence of various parameters such as the Pierce parameter and 1D gain parameter, the mechanism for the wide-band amplification is identified as the two-stream instability, which indeed has a much wider amplification band for electron beam with small energy spread.

While various authors have previously studied the two-stream instability and two-stream FEL[2-5], a self-consistent 3D model to describe the two-stream amplification process for a warm electron beam has not been fully developed, to our knowledge.

In this work, we started from the coupled Poisson-Vlasov equation system and derived an integral equation in the wave vector domain for the electron density variation induced by an arbitrary initial perturbation. Assuming the electrons have double-peak κ-2 velocity distribution, the integral equation reduces to a fourth order differential equation. For a point-like initial density perturbation, the solution has a close form in the wave vector domain, which is then inverse Fourier transformed to the configuration space using numerical method. In the second section, we solve the dispersion relation for an FEL with double-peak Lorentzian energy distribution and show that there is a wide-band growing mode when space charge is sufficiently strong. The growing rate is then compared with that of cold beam two-stream instability. The third section contains our derivation of the equation of motion and its general solution. We solve the initial value problem for a point-like initial perturbation in the fourth sectio and present numerical results of the electron density evolution induced by the perturbation. We summarize our studies in the last n.

* Work supported by Brookhaven Science Associates, LLC under Contract No.DE-AC02-98CH10886 with the U.S. Department of Energy.
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A WIDE-BAND GROWING SOLUTION IN FEL DISPERSION RELATION

The 1D FEL dispersion relation reads[6]

$$s = \left(1 + is\hat{\Lambda}_p^2\right) D(s), \tag{1}$$

where s is the Laplace transformation-variable of the normalized longitudinal location $\hat{z} \equiv \Gamma z$,

$$\hat{\Lambda}_p \equiv \frac{1}{\Gamma}\left[\frac{4\pi j_0}{\gamma_z^2 \gamma I_A}\right]^{1/2}, \tag{2}$$

is the space-charge parameter,

$$\Gamma \equiv \left[\frac{\pi j_0 \theta_s^2 \omega}{c\gamma_z^2 \gamma I_A}\right]^{1/3}, \tag{3}$$

is the 1D FEL gain parameter, $I_A \equiv m_e c^3/e$ is the Alfven current, ω is the radiation frequency, v_z is the longitudinal velocity of electrons, γ_z is the Lorentz parameter for v_z,

$$\hat{\Delta} \equiv -\frac{1}{\Gamma}\left[k_w + \frac{\omega}{c} - \frac{\omega}{v_z}\right] \tag{4}$$

is the normalized detuning parameter, $k_w = 2\pi/\lambda_w$ is the undulator's wave number,

$$\rho = \gamma_z^2 \Gamma c/\omega \tag{5}$$

is the Pierce parameter. The dispersion integral in eq. (1) is defined as

$$D(s) \equiv \int_{-\infty}^{\infty} d\hat{P} \frac{d\hat{F}(\hat{P})}{d\hat{P}} \frac{1}{s + i(\hat{P} - \hat{\Delta})}, \tag{6}$$

for any root of eq. (1) with $\mathrm{Re}(s) > 0$ to correspond to an exponential growing FEL instability. Taking the energy distribution as

$$\hat{F}(\hat{P}) = \frac{1}{2\pi\sigma}\left[\frac{1}{1 + \left(\hat{P}/\sigma - \xi\right)^2} + \frac{1}{1 + \left(\hat{P}/\sigma + \xi\right)^2}\right], \tag{7}$$

and inserting it into eq. (6) yields

$$D(s) = i\frac{\left(s + \sigma - i\hat{\Delta}\right)^2 - (\zeta\sigma)^2}{\left[\left(s + \sigma - i\hat{\Delta}\right)^2 + (\zeta\sigma)^2\right]^2}, \tag{8}$$

which combines with eq. (1) leads to the following two-stream FEL dispersion relation:

ISBN 978-3-95450-126-7

$$s\left[\left(s+\sigma-i\hat{\Delta}\right)^2+(\zeta\sigma)^2\right]^2 = i\left[\left(s+\sigma-i\hat{\Delta}\right)^2-(\zeta\sigma)^2\right]\left(1+is\hat{\Lambda}_p^2\right) . \quad (9)$$

As defined in eq. (3) and (5), Pierce parameter and 1D gain parameter are function of the radiation frequency, or detuning. Since FEL instability is typically narrow-band, these two parameters are often Taylor expanded around FEL resonant frequency and only lowest order values are kept. However, when the amplification band is wide, the expansion is invalid for frequencies far away from the resonant frequency. Figure 1 shows the growing solution of the dispersion relation, eq. , for parameters listed in Table 1. The growth rate of two-stream instability dispersion relation for cold electrons reads

$$\left[\mathrm{Im}(\omega)\right]^2 = -\omega_{p0}^2\left(\frac{1+2y^2-\sqrt{1+8y^2}}{2}\right) , \quad (10)$$

with

$$y=\frac{(v_1-v_2)k}{2\omega_{p0}} . \quad (11)$$

As shown in Fig. 1, the growth rate solved from the FEL dispersion relation, eq. (1), exactly overlaps with the two-stream instability growth rate except for a small range around the FEL resonant frequency, which suggests that the wide-band growing solution of the FEL dispersion relation originates from the two stream instability.

Table 1: Parameters used in Generating Figure 1

Peak current	100 A
Electron energy	20 MeV
Wiggler period	1 cm
Wiggler parameter, a_w	0.2
Energy speration	40 KeV

ANALYTICAL MODEL FOR TWO-STREAM INSTABILITIES

The couple Vlasov-Poisson equation system describes the collision-less electron plasma, which in the co-moving beam frame reads

$$\frac{\partial}{\partial t}f_1(\vec{x},\vec{v},t)+\vec{v}\cdot\frac{\partial}{\partial\vec{x}}f_1(\vec{x},\vec{v},t)-\frac{\vec{E}(\vec{x},t)e}{m_e}\cdot\frac{\partial}{\partial\vec{v}}f_0(\vec{v})=0 , \quad (12)$$

and

$$\nabla^2\varphi(\vec{x},t)=-\frac{e}{\varepsilon_0}n_1(\vec{x},t) , \quad (13)$$

with

$$n_1(\vec{x},t)=\int_{-\infty}^{\infty}f_1(\vec{x},\vec{v},t)d^3v , \quad (14)$$

and

$$\vec{E}(\vec{x},t)=-\vec{\nabla}\varphi(\vec{x},t) . \quad (15)$$

Fourier transform eq. (12-(15) to the wave vector domain yields

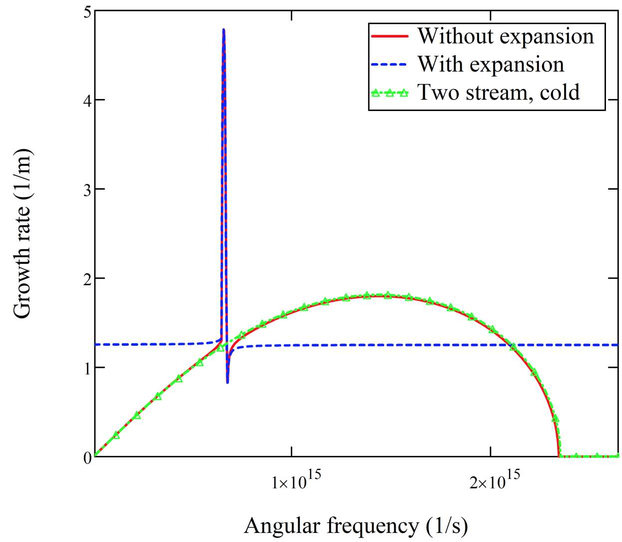

Figure 1: The growing root of the 1D FEL dispersion relation for two-stream cold beam as calculated from eq (9). The blue dash curve is the growing root of eq. (9) calculated with the values of Pierce parameter and 1D gain parameter taken at the resonant frequency. The red solid curve is the growing root of eq. (9) calculated with the frequency dependent Pierce parameter and 1D gain parameter. The green triangles is the growing rate for cold beam two stream instabilities.

$$\tilde{f}_1(\vec{k},\vec{v},t)=\tilde{f}_1(\vec{k},\vec{v},0)e^{-i\vec{k}\cdot\vec{v}t} +i\frac{\omega_p^2}{k^2}\int_0^t\tilde{n}_1(\vec{k},t_1)e^{i\vec{k}\cdot\vec{v}(t_1-t)}\vec{k}\cdot\frac{\partial}{\partial\vec{v}}f_0(\vec{v})dt_1 , \quad (16)$$

with

$$\tilde{f}_1(\vec{k},\vec{v},t)=\int_{-\infty}^{\infty}\tilde{f}_1(\vec{x},\vec{v},t)e^{-i\vec{k}\cdot\vec{x}}d^3x , \quad (17)$$

and

$$\tilde{n}_1(\vec{k},t)=\int_{-\infty}^{\infty}\tilde{n}_1(\vec{x},t)e^{-i\vec{k}\cdot\vec{x}}d^3x . \quad (18)$$

Integrating eq. (16) over the velocities leads to the following integral equation:

$$\tilde{n}_1(\vec{k},t)=\int_{-\infty}^{\infty}\tilde{f}_1(\vec{k},\vec{v},0)e^{-i\vec{k}\cdot\vec{v}t}d^3v +\omega_p^2\int_0^t\tilde{n}_1(\vec{k},t_1)(t_1-t)g_0\left(\vec{k}(t_1-t)\right)dt_1 , \quad (19)$$

where

$$g(\vec{u})=\frac{1}{n_0}\int_{-\infty}^{\infty}f_0(\vec{v})e^{-i\vec{u}\cdot\vec{v}}d^3v . \quad (20)$$

For simplicity, we take un-perturbed velocity distribution of the electrons as

ISBN 978-3-95450-126-7

$$f_0(\vec{v}) = \frac{n_0}{2\pi^2\beta_x\beta_y\beta_z}\left\{\left[1+\frac{v_x^2}{\beta_x^2}+\frac{v_y^2}{\beta_y^2}+\frac{(v_z+v_{sz})^2}{\beta_z^2}\right]^{-2} + \left[1+\frac{v_x^2}{\beta_x^2}+\frac{v_y^2}{\beta_y^2}+\frac{(v_z-v_{sz})^2}{\beta_z^2}\right]^{-2}\right\}, \quad (21)$$

where $2v_{sz}$ is the velocity separation of the two streams in a frame in which the average velocity of all electrons is zero. Inserting eq. (21) into eq. (20) leads to

$$g(\vec{u}) = \exp[-R(\vec{u})]\cos(u_z v_{sz}), \quad (22)$$

with

$$R(\vec{u}) = \sqrt{(u_x\beta_x)^2+(u_y\beta_y)^2+(u_z\beta_z)^2}. \quad (23)$$

Inserting eq. (22) into eq. (19) produces

$$\tilde{H}_1(\vec{k},t) = e^{-\lambda(\vec{k})t}\int_{-\infty}^{\infty}\tilde{f}_1(\vec{k},\vec{v},0)e^{-i\vec{k}\cdot\vec{v}t}d^3v + \omega_p^2\int_0^t \tilde{H}_1(\vec{k},t_1)(t_1-t)\cos[k_z v_{sz}(t-t_1)]dt_1, \quad (24)$$

with

$$\tilde{H}_1(\vec{k},t) \equiv \tilde{n}_1(\vec{k},t)e^{-\lambda(\vec{k})t}, \quad (25)$$

and

$$\lambda(\vec{k}) \equiv -\sqrt{(k_x\beta_x)^2+(k_y\beta_y)^2+(k_z\beta_z)^2}. \quad (26)$$

Taking the fourth derivative of eq. (24) homogeneous ODE:

$$\left[\frac{d^4}{dt^4}+(2k_z^2v_{sz}^2+\omega_p^2)\frac{d^2}{dt^2}+k_z^2v_{sz}^2(k_z^2v_{sz}^2-\omega_p^2)\right]\tilde{H}_1(\vec{k},t) = \frac{d^4}{dt^4}\left[e^{-\lambda(\vec{k})t}\int_{-\infty}^{\infty}\tilde{f}_1(\vec{k},\vec{v},0)e^{-i\vec{k}\cdot\vec{v}t}d^3v\right]. \quad (27)$$

The behaviour of the system is determined by the eigenvalues of eq. (27), which can be solved as

$$\alpha_{1,2} = \pm\frac{\omega_p}{\sqrt{2}}\left(\sqrt{1+\frac{8k_z^2v_{sz}^2}{\omega_p^2}}-1-\frac{2k_z^2v_{sz}^2}{\omega_p^2}\right)^{\frac{1}{2}}, \quad (28)$$

and

$$\alpha_{3,4} = \pm i\frac{\omega_p}{\sqrt{2}}\left(\sqrt{1+\frac{8k_z^2v_{sz}^2}{\omega_p^2}}+1+\frac{2k_z^2v_{sz}^2}{\omega_p^2}\right)^{\frac{1}{2}}. \quad (29)$$

The first eigenvalue,

$$\alpha_1 = \frac{\omega_p}{\sqrt{2}}\left(\sqrt{1+\frac{8k_z^2v_{sz}^2}{\omega_p^2}}-1-\frac{2k_z^2v_{sz}^2}{\omega_p^2}\right)^{\frac{1}{2}}, \quad (30)$$

is identical to the growth rate of a cold beam two-stream instability, eq. (10). However, eq. (25) implies that there is an additional Landau damping term in addition to the exponential growing factor due to the two-stream instability.

EXAMPLE FOR POINT LIKE INITIAL PERTURBATION

As an example of applying the formalism developed in the second section, we consider the following initial perturbation:

$$f_1(\vec{x},\vec{v},0) = \frac{\delta(\vec{x})}{\pi^2\beta_x\beta_y\beta_z}\left[1+\frac{v_x^{\,2}}{\beta_x^2}+\frac{v_y^{\,2}}{\beta_y^2}+\frac{v_z^{\,2}}{\beta_z^2}\right]^{-2}. \quad (31)$$

The velocity distribution of the initial perturbation in eq. (31) is specifically chosen such that the inhomogeneous driving term in eq. (27) vanishes. The general solution for the resulting homogenous part of the differential equation reads

$$\tilde{H}_1(\vec{k},t) = \sum_{i=1}^{4}B_i(\vec{k})e^{\alpha_i(k_z)t}, \quad (32)$$

with $B_i(\vec{k})$ being coefficients to be determined by the initial density perturbation and its derivatives. Making use of eq. (25), we obtain the solution for the electron density

$$\tilde{n}_1(\vec{k},t) = \sum_{i=1}^{4}B_i(\vec{k})e^{[\alpha_i(k_z)+\lambda(\vec{k})]t}. \quad (33)$$

There is only one possible growing term out of the four terms in the summation of eq. (33) with growth rate

$$\Gamma_{grow} = \alpha_1(k_z)+\lambda(\vec{k}). \quad (34)$$

For given longitudinal wave vector, k_z, the optimal velocity separation of the two streams is

$$v_{sz} = \sqrt{\frac{3}{8}}\frac{\omega_p}{k_z}, \quad (35)$$

with the maximal growth rate of

$$\Gamma_{\max} = \frac{\omega_p}{2\sqrt{2}} - k_z\beta_z. \quad (36)$$

Eq. (36) suggests that there is a short wavelength limit,

$$\lambda_{z,\min} = 4\pi\sqrt{2}\frac{\beta_z}{\omega_p} \approx 17.7\frac{\beta_z}{\omega_p}, \quad (37)$$

and the two-stream instability does not amplify electron density modulations with wavelength shorter than $\lambda_{z,\min}$.

The initial electron density and its derivatives are given by the initial phase space density perturbation, eq. (31), which, in the wave vector domain, reads

$$\tilde{n}_1(\vec{k},0) = 1, \quad (38)$$

and

$$\left.\frac{d^{(n)}}{dt^{(n)}}\tilde{n}_1(\vec{k},t)\right|_{t=0} = 0, \quad (39)$$

with $n=1,2,3$. Applying the initial condition of eq. (38) and (39), the coefficients $B_i(\vec{k})$ are determined as follows:

$$B_1 = \frac{(\alpha_3^2-\lambda^2)(\alpha_1-\lambda)}{2\alpha_1(\alpha_3^2-\alpha_1^2)}, \quad (40)$$

$$B_2 = \frac{(\alpha_3^2-\lambda^2)(\alpha_1+\lambda)}{2\alpha_1(\alpha_3^2-\alpha_1^2)}, \quad (41)$$

$$B_3 = \frac{\left(\alpha_1^2 - \lambda^2\right)\left(\alpha_3 - \lambda\right)}{2\alpha_3\left(\alpha_1^2 - \alpha_3^2\right)} \ , \qquad (42)$$

and

$$B_4 = \frac{\left(\alpha_1^2 - \lambda^2\right)\left(\alpha_3 + \lambda\right)}{2\alpha_3\left(\alpha_1^2 - \alpha_3^2\right)} . \qquad (43)$$

The electron density in the configuration space is then given by inverse Fourier transformation of eq. (33). Assuming the velocity spread in the transverse plane are the same, i.e.

$$\beta_x = \beta_y = \beta_\perp \ , \qquad (44)$$

the 3D inverse Fourier transformation of eq. (33 can be expressed as

$$\begin{aligned} n_1(\vec{x},t) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \tilde{n}_1\left(\vec{k},t\right) e^{i\vec{k}\cdot\vec{x}} d^3k \\ &= \frac{1}{2(2\pi)^2}\int_{-\infty}^{\infty} dk_z e^{ik_z z}\int_0^{\infty} J_0\left(k_\perp r_\perp\right)\tilde{n}_1\left(k_z,k_\perp^2,t\right)dk_\perp^2 \end{aligned} \qquad (45)$$

We used numerical approach to evaluate the evolution of the electron density as predicted by eq. (45) for $v_{sz} = 4\beta_z$. Figure 2 shows a numerical calculation of the growth rate as a function of the longitudinal wave vector, k_z, for $k_x = k_y = 0$. Figure 3 shows the time evolution of the electron density variation along the longitudinal location of the beam for a specific transverse location. The evolution is dominated by damping of the initial modulation amplitude both due to the motion of the streams and Landau damping from the velocity spread. After about two plasma oscillation, the two-stream instability takes over and a wave-packet with wavelength about 50 Debye length starts to develop.

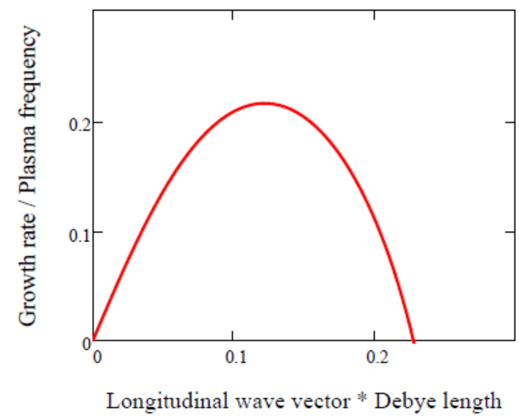

Figure 2: The growth rate of two-stream instability as calculated from eq. (34) with $k_x = k_y = 0$. The abscissa is the longitudinal wave vector multiplied by the longitudinal Debye length, i.e. $k_z\beta_z/\omega_p$. The ordinate is the growth rate in unit of angular plasma frequency, ω_p.

The amplitude of the wave-packets continues growing for the rest of the time with a growth rate about three fold per plasma period. Figure 4 shows the 2D contour plots of density modulation for identical parameters used in generating fig. 3, which shows that the transverse area of the wave-packet is about 100 transverse Debye radius and the longitudinal width is about 200 longitudinal Debye radius after 10 plasma oscillations.

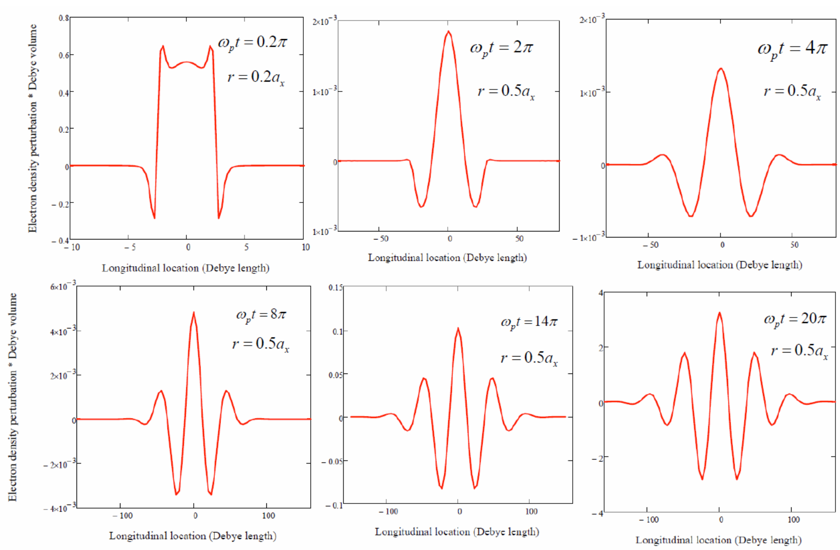

Figure 3: Evolution of electron density in the configuration space as calculated from eq. (45). The abscissas of the plots are the longitudinal location along the electron beam in units of longitudinal Debye length, $a_z = \beta_z/\omega_p$ and the ordinates are the electron density variation in units of $1/\left(a_\perp^2 a_z\right)$, with $a_\perp = \beta_\perp/\omega_p$ being the transverse Debye radius. Each of the six snapshot is taken at a given time and transverse location. The transverse locations are specified in the plots by $r = \sqrt{x^2+y^2}$, in units of the transverse Debye radius. The time that these snapshots are taken is specified by the plasma phase advances, $\omega_p t$. The longitudinal velocity separation of the two streams is $v_{sz} = 4\beta_z$ for all plots.

ISBN 978-3-95450-126-7

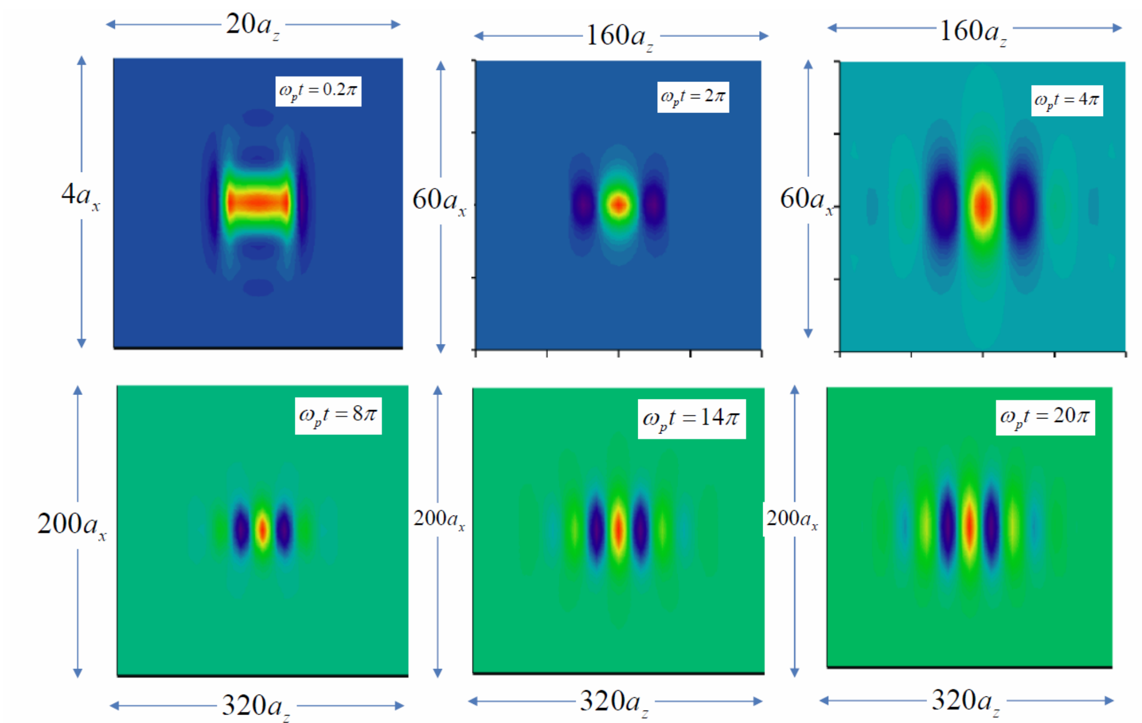

Figure 4: Contour plots of Fig. 3 showing the 2D images of the density modulation. The horizontal axis is the longitudinal location along the direction of the stream velocity and the vertical axis is the transverse location perpendicular to the direction of the stream velocity. The spans of the plotted spatial ranges and the time when the snapshots are taken are specified in the plots.

SUMMARY

In this work, we developed an analytical model to study two stream instabilities for a warm electron beam with double peak Lorentzian velocity distribution. The Vlasov-Poisson system is reduced to a fourth order inhomogeneous ordinary differential equation (ODE) in the wave vector domain, which, in general, can be solved for arbitrary initial phase space initial perturbation. As a simple example, the differential equation is solved for a specifically chosen initial phase space density perturbation, which allows the inhomogeneous driving term to vanish. The wave vector domain solution is then numerically inverse Fourier transformed into the configuration space and the resulting electron density evolution in a two-stream electron plasma is presented.

According to our model, the two stream instability will not amplify electron density modulation with wavelength below 17.7 longitudinal Debye radius.

ACKNOWLEDGEMENT

Authors would like to thank M. Blaskiewicz for helping with many techniques used in developing this model.

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ISBN 978-3-95450-126-7