EFFECT OF COULOMB COLLISIONS ON ECHO-ENABLED HARMONIC GENERATION*

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Abstract

We develop a practical computational technique for evaluation of the effect of intra-beam scattering on Echo-Enabled Harmonic Generation (EEHG). The techniques is applied for calculation of the EEHG seeding for NGLS soft x-ray FEL project being developed at LBNL.

INTRODUCTION

Echo-Enabled Harmonic Generation [1,2] (EEHG) has a remarkable up-frequency conversion efficiency and allows for generation of high harmonics with a relatively small energy modulation. While increasing harmonic number to $\sim 10^2$ in EEHG sets stringent requirements on the seeding system, they can, in principle, be satisfied with increased tolerances on the magnetic field and the quality of the laser beams used for modulation of the beam energy. It was recently realized [3] however that the ultimate limit on harmonic number in EEHG is likely imposed by Coulomb collisions between the particles of the beam (aka intra-beam scattering). This is due to the fact that in the process of EEHG the phase space of the beam is split into stripes that have effective energy spread of the order of the energy spread of the beam divided by the harmonic number, that is much smaller than the beam energy spread. As is well known, the dominant process in Coulomb collisions is a small-angle scattering, which predominantly leads to diffusion in the momentum space. This diffusion smears out the stripes and eventually lead to decreasing of the final bunching factor at the desired harmonic.

The analysis in [3] was limited by an unrealistic assumption that collisions occur in a beam with a constant transverse size and angular spread. In this paper we will get rid of this constraint and consider intra-beam scattering in a lattice where the transverse size of the beam and its angular spread vary along the beam path. Account of these effects leads to the energy diffusion in the beam varying with distance. A convenient technique for treating such a diffusion was developed in recent papers by Yampolsky and Carlsten [4, 5] and is based on Fourier transformation of the beam distribution function in six dimensional phase space and a subsequent solution of the Fourier transformed Vlasov equation. In this paper we briefly outline their technique before applying it to the particular example of EEHG. The technique is applicable for linear beam dynamics when collective effects (wakefields) are neglected.

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In a recent report [6] G. Penn developed an alternative approach to the IBS in EEHG and carried out computer simulations for several layouts of EEHG seeding for the NGLS soft x-ray FEL project at LBNL [7].

COULOMB COLLISIONS IN EEHG

A simplified collision term which can be used in EEHG was derived in [3]. It appears on the right-hand side of the Vlasov equation for the distribution function f, and can be written in the following form

$$\frac{df}{ds} = \frac{1}{2} D(s) \frac{\partial^2 f}{\partial \Delta E^2}, \tag{1}$$

where the full derivative df/ds in (1) is taken along particles' trajectories, s is the path length and ΔE is the energy measured from the nominal energy of the beam. The diffusion coefficient D is

$$D(s) = \frac{\pi^{1/2}\Lambda}{2\gamma\sqrt{\sigma_{\theta x}(s)\sigma_{\theta y}(s)}} \frac{(m_e c^2)^2 r_e}{\sigma_x(s)\sigma_y(s)} \frac{I}{I_A}, \tag{2}$$

with I the beam current, $I_A = mc^3/e \approx 17$ kA the Alfven current and r_e the classical electron radius. The diffusion coefficient is averaged over the transverse distribution of the beam, which is assumed to be a round Gaussian with the rms transverse sizes σ_x and σ_y and the rms angular spreads $\sigma_{\theta x}$ and $\sigma_{\theta y}$ in x and y directions, respectively.

In Eq. (2) we specifically indicate that the beam dimensions and divergences vary with s due to the variation of the lattice functions. Analysis of Ref. [3] neglected this variation and assumed D constant.

EVOLUTION OF THE DISTRIBUTION FUNCTION IN FOURIER SPACE

The distribution function of the beam f in the six-dimensional phase space, $f(x, \theta_x, y, \theta_y, z, \Delta E, s)$, satisfies the Vlasov equation (we define f as a probability in the phase space, so that its integration over the first 6 variables gives unity). The arguments of f are the transverse coordinates x and y, the transverse angles θ_x and θ_y, the longitudinal coordinate inside the beam z, and the energy deviation from the nominal energy ΔE. If there is no interaction between the particles, so that one actually deals with one-particle dynamics, a solution of the Vlasov equation can be obtained with the help of an R-matrix.

Let $\boldsymbol{X} = (x, \theta_x, y, \theta_y, z, \Delta E)^T$ be a column vector; if the 6×6 matrix $R(s)$ that transforms this vector from $s = 0$ to s is known, then $\boldsymbol{X}(s) = R(s)\boldsymbol{X}(0)$. With the

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help of R the distribution function $f(\boldsymbol{X}, s)$ at location s is easily expressed through the initial distribution at $s = 0$, $f_0(\boldsymbol{X}) = f(\boldsymbol{X}, 0)$, as

$$f(\boldsymbol{X}, s) = f_0(R(s)^{-1}\boldsymbol{X}). \tag{3}$$

If we now make a 6-dimensional Fourier transformation $f \to \hat{f}$,

$$\hat{f}(k_x, k_{\theta x}, k_y, k_{\theta y}, k_z, k_{\Delta E}, s) = \int dx d\theta_x dy d\theta_y dz d\Delta E \times e^{i(xk_x+\theta_x k_{\theta x}+yk_y+\theta_y k_{\theta y}+zk_z+\Delta E k_{\Delta E})} f,$$

then as is shown in [4, 5] $\hat{f}$ also satisfies a (transformed) Vlasov equation whose solution can be found similar to (3). Let $\boldsymbol{K}$ denote the column vector $\boldsymbol{K} = (k_x, k_{\theta x}, k_y, k_{\theta y}, k_z, k_{\Delta E})^T$, then the Fourier transformed distribution function $\hat{f}(\boldsymbol{K}, s)$ at s is expressed through the initial $\hat{f}_0(\boldsymbol{K}) = \hat{f}(\boldsymbol{K}, 0)$ via the transposed matrix R^T [4,5] ,

$$\hat{f}(\boldsymbol{K}, s) = \hat{f}_0(R(s)^T\boldsymbol{K}). \tag{4}$$

Let us now consider the 6 numbers $(k_x, k_{\theta x}, k_y, k_{\theta y}, k_z, k_{\Delta E})$ as a point in a 6-dimensional Fourier phase space. When the beam is moving through a lattice, each such phase point is moving in the Fourier phase space according to

$$\boldsymbol{K}(s) = \left(R(s)^T\right)^{-1}\boldsymbol{K}(0), \tag{5}$$

and carries the value of $\hat{f}$ with it. In analogy with real particles characterized by vectors $\boldsymbol{X}$ it makes sense to associate with the vector $\boldsymbol{K}$ a *quasi-particle* which is characterized by the six wavenumbers along the six axes of the Fourier space[1].

Note that the quantity

$$b(k_z, s) = \hat{f}(0,0,0,0,k_z,0,s) = \int dx d\theta_x dy d\theta_y dz d\Delta E e^{izk_z} f \tag{6}$$

is the Fourier transform of the longitudinal density of the bunch. It is equal to the often defined *bunching factor* $N^{-1}\sum_{j=1}^{N} e^{ik_z z_j}$, where z_j is the z-coordinate of particle j and N is the number of particles in the beam. If a beam is sent through a radiator, the intensity of coherent radiation at frequency $\omega = ck_z$ will be proportional to $|Qb(k_z)|^2$, where Q is the bunch charge.

DIFFUSION IN THE VLASOV EQUATION

With account of the energy diffusion (1) function $\hat{f}$ satisfies the Fourier transformed Vlasov equation

$$\frac{d\hat{f}}{ds} = -\frac{1}{2}Dk_{\Delta E}^2\hat{f}. \tag{7}$$

[1] In analogy with quasi-particles in different branches of physics (*e.g.*, phonons, plasmons, magnons, *etc.*) we can call these quasi-particles *bunchions*, to reflect the fact they characterize bunching of real particles in the phase space at difference wavelength.

Comparing it with (1) we see that the right hand side is not a differential operator any more. This equation can be easily integrated,

$$\hat{f}(\boldsymbol{K}, s) = \exp\left[-\frac{1}{2}\int_0^s D(s')k_{\Delta E}(s')^2 ds'\right] \times \hat{f}_0(R(s)^T\boldsymbol{K}), \tag{8}$$

where $k_{\Delta E}(s)$ is the $k_{\Delta E}$ coordinate of the quasi-particle from initial to final state (it is given by the sixth element of the vector $\boldsymbol{K}(s)$ in (5)).

FOURIER ANALYSIS OF EEHG

We will now illustrate the general technique outlined in the previous sections by applying it to EEHG seeding [1, 2]. In this case we deal with the longitudinal dynamics only, and limit our consideration to two variables z and $p = \Delta E/\sigma_E$ with σ_E the rms energy spread of the beam.

While analysis of the previous sections takes into account the finite length of the bunch, here we will consider the case of infinitely long bunch with a flat profile, assuming the initial distribution function of the form

$$f_0(p, z) = \frac{1}{\sqrt{2\pi}L}e^{-p^2/2}, \tag{9}$$

for $-L/2 < z < L/2$ and $f_0 = 0$ outside of this interval. Then, in the limit $L \to \infty$,

$$\hat{f}_0(k_p, k_z) = \int_{-L/2}^{L/2} dz \int_{-\infty}^{\infty} dp f_0(p, z)e^{ik_p p+ik_z z} \to 2\pi L^{-1}e^{-k_p^2/2}\delta(k_z), \tag{10}$$

where we now use the variable $k_p = k_{\Delta E}\sigma_E$ associated with p instead of $k_{\Delta E}$.

We now consider an energy modulation in the first undulator. It changes particle's energy from p to p', $p' = p + A_1\sin(k_1 z)$, where $A = \Delta E_1/\sigma_E$ with ΔE_1 the energy modulation amplitude and ck_1 is the frequency of the first laser beam. This transforms the distribution function from f_0 to f_1, $f_1(p, z) = f_0(p - A_1\sin(k_1 z), z)$. In Fourier space we have

$$\hat{f}_1(k_p, k_z) = \int dz dp f_0(p - A_1\sin(k_1 z), z)e^{ik_p p+ik_z z}.$$

Changing the integration variable from p to $p - A_1\sin(k_1 z)$ and using the expansion $e^{ik_p A_1\sin(k_1 z)} = \sum_{m=-\infty}^{\infty} e^{-imk_1 z}J_{-m}(A_1 k_p)$, we obtain

$$\hat{f}_1(k_p, k_z) = \frac{2\pi}{L}\sum_{m=-\infty}^{\infty} J_{-m}(A_1 k_p)\, e^{-k_p^2/2}\delta(k_z - mk_1). \tag{11}$$

We see that as the result of the energy modulation each Fourier quasi-particle (k_z, k_p) *decays* (with the relative amplitudes given by $J_{-m}(A_1 k_p)$) into infinitely many new particles with $(k_z + k_1 m, k_p)$, $m = 0, \pm1, \pm2, \ldots$, shifted

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horizontally in the Fourier phase space. Strictly speaking, the energy modulation in an undulator is a nonlinear process and is not described by the formalism of R-matrix. However, given that the component k_p (and hence $k_{\Delta E}$) in the decay does not change, we still can use (8) multiplying the decay amplitude $J_{-m}(A_1 k_p)$ by the factor $\exp\left(-\frac{1}{2}L_u D k_{\Delta E}^2\right)$, where L_u is the length of the undulator and D is the diffusion coefficient in it (assumed constant through the undulator).

The chicane after the first modulator with the $R_{56}^{(1)}$ dispersion strength shifts the particles along z, $z' = z + C_1 p$, where $C_1 = R_{56}^{(1)}\sigma_E/E_0$, and transforms $f_1(p,z)$ into $f_1(p, z - C_1 p)$. The Fourier transform of the distribution function after the chicane is

$$\hat{f}_2(k_p, k_z) = \int dzdp f_1(p, z - C_1 p) e^{ik_p p + ik_z z} = \hat{f}_1(k_p + k_z C_1, k_z). \quad (12)$$

The calculations outlined above can be repeated for the second stage of EEHG and combined together give the final distribution function $\hat{f}_4$ in the Fourier phase space:

$$\hat{f}_4(k_p,k_z) = \frac{2\pi}{L} \sum_{l,m=-\infty}^{\infty} J_{-l}\left(A_2[k_p + (lk_2 + mk_1)C_2]\right) \times J_{-m}\left(A_1[(k_p + (lk_2 + mk_1)C_2) + C_1 mk_1]\right) \times e^{-([k_p + (lk_2+mk_1)C_2] + C_1 mk_1)^2/2} \times \delta(k_z - lk_2 - mk_1), \quad (13)$$

where $A_2 = \Delta E_2/\sigma_E$ with ΔE_2 the amplitude of the energy modulation in the second undulator, ck_2 is the frequency of the second laser beam and $C_2 = R_{56}^{(2)}\sigma_E/E_0$ where $R_{56}^{(2)}$ is the dispersion strength of the second modulator. If we now set $k_p = 0$ in this equation (see Eq. (6)), find the coefficients in front of the delta functions $\delta(k_z - lk_2 - mk_1)$ and multiply them by $(2\pi)^{-1}L$, we obtain the bunching factors $b_{l,m}$ for various EEHG harmonics as defined in [2].

PHASE SPACE ILLUSTRATIONS

The analysis of the previous section is illustrated in Fig. 1 by Fourier phase space pictures for the evolution of the distribution function $\hat{f}$. In a particular example of this section we choose the EEHG parameters to generate the harmonic $h = 200$ of the laser frequency (we assume $k_1 = k_2 \equiv k_L$). While equations of the previous section assume $L \to \infty$ and hence $\hat{f}_0 \propto \delta(k_z)$ for illustration purposes we consider the bunch of finite length (to have it well resolved in density plots we actually take an unrealistically short bunch, $\sigma_z \approx 3\lambda_L$). In these illustrations we neglect the diffusion effect (8).

The initial distribution function is shown on the first picture of Fig. 1—it is a two-dimensional Gaussian localized at the origin $k_p \approx k_z \approx 0$. The second figure shows the Fourier phase space after the first undulator where the initial function is split in the horizontal direction according to Eq. (11). The third figure shows a small part of the phase space after the first chicane; note that the chicane has shifted the dense (red) areas in the phase space vertically to $k_p \approx 30$. The fourth picture shows the previous area moved after the second undulator-modulator to the desired harmonic $h = 200$. However, it still has the same $k_p \approx 30$, and hence does not contribute to the density modulation of the beam. The final, fifth, picture shows the relevant phase area after the second chicane: it is now shifted in vertical direction on the horizontal axis $k_p = 0$. According to Eq. (6) this area is now responsible for the bunching factor with $h = 200$.

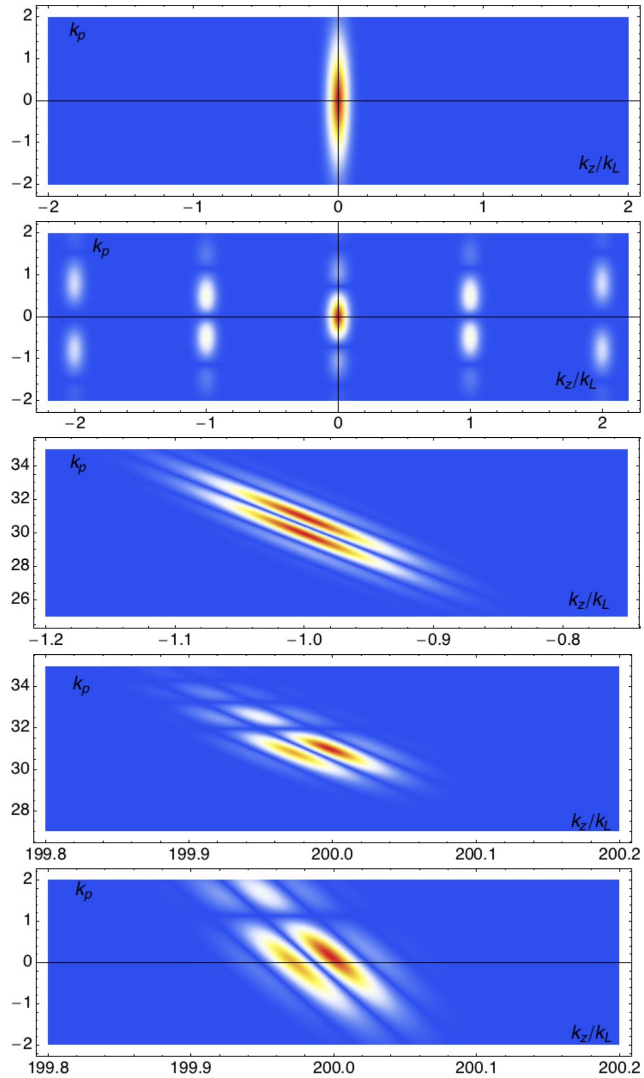

Figure 1: Sequence of phase plots of the absolute value of $|\hat{f}|$ through the EEHG system. The horizontal coordinate is k_z/k_L and the vertical one is k_p.

PRACTICAL EXAMPLE OF EEHG FOR NGLS

As a practical example we calculate the effect of IBS using EEHG lattice for NGLS [7] using parameters of the

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seeding scheme provided to the author by G. Penn. The relevant parameters of the beam and the laser are listed in Table 1. The seeding scheme is aimed at generation of 1 nm

Table 1: Representative Set of NGLS Parameters

Electron beam energy	2.4 GeV
Bunch peak current	600 A
Normalized emittance	0.6 μm
Energy spread, σ_E	150 keV
Laser wavelength	200 nm
First/second energy modulation	0.5/1 MeV
Seed wavelength	1 nm

seed wavelength using the laser wavelength $\lambda_L = 200$ nm, that is generation of harmonic 200 of the laser radiation. It uses two undulator modulators each 2 m long. The first energy modulation is 0.5 MeV, and the second energy modulation is 1 MeV. The first chicane has $R_{56}^{(1)} = 15.5$ mm and the second one has $R^{(2)} = 79$ micron. Neglecting the diffusion in the beam, for the optimized EEHG parameters, the theory [2] predicts the bunching factor at harmonic 200 approximately equal to 5%, $b_{200} \approx 0.05$. Plots of the beta functions and the R_{56} function through the EEHG system are shown in Figs. 2 and 3. Note that the variation of R_{56}

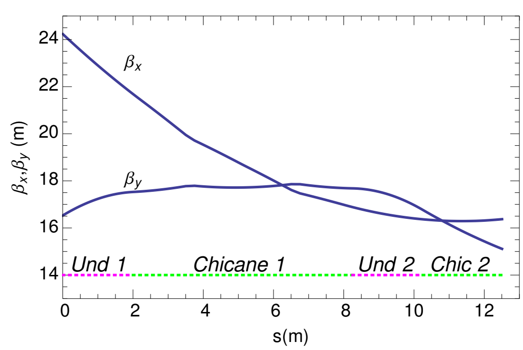

Figure 2: Plots of the horizontal and vertical beta functions in the seeding area. The color dashed lines at the bottom (here and at the next two figures) show locations of the two undulators and two chicanes.

in the second chicane is too small to be visible in Fig. 3.

Taking the Coulomb logarith $\Lambda = 8$ and using the lattice functions from Fig. 2 (and also the dispersion and its derivative which we do not show) we calculated the diffusion coefficients (2). The plot of $D(s)$ is shown in Fig. 4 and the product $k_{\Delta E}^2 D$ that enters Eq. (8) is shown in Fig. 5. Note that this product starts to grow from the middle of the first chicane where R_{56} increases and the phase space of the beam develops thin energy stripes in the phase space of the beam. Integration of this product through the system gives the suppression factor

$$e^{-(1/2)\int_0^s D(s) k_{\Delta E}(s)^2 ds} = 0.47,$$

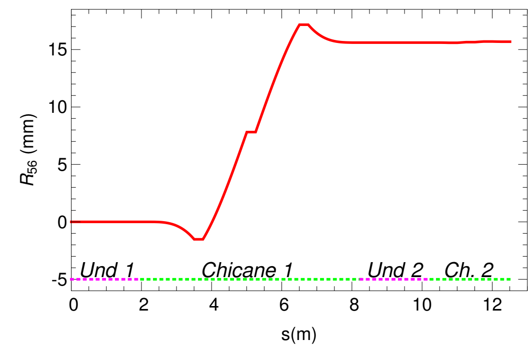

Figure 3: Plot of $R_{56}(s)$ in EEHG seeding for NGLS.

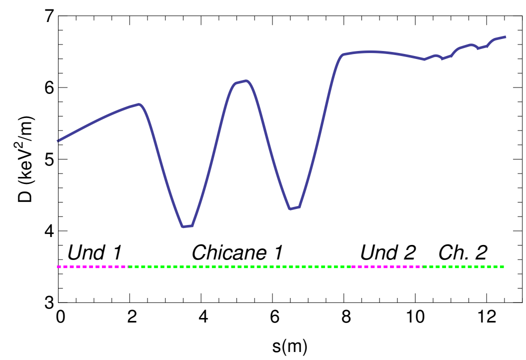

Figure 4: Diffusion coefficient as a function of s.

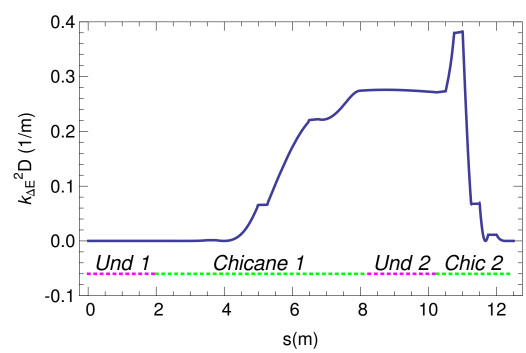

Figure 5: Plot of the product $k_{\Delta E}^2 D$.

and reduces its value from 5% to $\approx 2.5\%$.

DISCUSSION

In addition to IBS studied in this paper there is another diffusion mechanism that adds to the intra-beam scattering—quantum diffusion due to the incoherent synchrotron radiation. This diffusion can be straightforwardly added to (1) and included in the analysis of the seeding. Given that the quantum diffusion scales as B^3 with the magnetic field B, an optimized design should try to lower

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the magnetic field in the second undulator where this diffusion is likely to be dominant.

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