TRANSVERSE INSTABILITIES OF COASTING BEAMS WITH SPACE CHARGE*

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Abstract

To analyze transverse stability of beams with significant space charge, a rigid-beam model is usually used. In this paper the validity of this model is considered. It is concluded that the model is valid for a relatively small area of parameters which, however, is the most interesting for practical applications. Then the model is used for derivation of Landau damping rate in a general case. The results are applied to a round Gaussian beam. Its stability threshold is described by simple fits for the cases of chromatic and octupole tune spreads.

INTRODUCTION

Beam particles interact with each other through walls of the vacuum chamber. This interaction is conventionally described in terms of the wake functions and impedances. Generally, the wake fields lead to beam coherent instabilities. However, if a beam frequency spread is sufficiently large, there are particles that stay in resonance with coherent motion. As a result, if their density is sufficiently large, the instability is stabilized. This dissipation mechanism is called the Landau damping. Its damping rate is proportional to the phase space density of resonant particles. Contrary to the wake fields, Coulomb interaction does not drive the instability by itself, since it preserves energy and momentum. However, the collective Coulomb field can strongly affect beam stability because it separates coherent and incoherent frequencies. Indeed, when the beam oscillates as a whole, its collective motion does not see the space charge, while an individual particle oscillation does. Thus, if the coherent and incoherent frequencies are separated, there are no resonant particles and consequently no Landau damping, resulting in beam instability.

To analyze beam stability with the space charge, an effective method was presented in 1974 by D. Möhl and H. Schönauer [1]. To describe transverse oscillations of a coasting beam, a heuristic equation of motion was suggested:

$$\frac{d^2 x_i}{dt^2} + \Omega_i^{\,2} Q_i^{\,2} x_i + 2\Omega_0^{\,2} Q_0 \left(\Delta Q_c \bar{x} + \Delta Q_{sc}\left(x_i - \bar{x}\right)\right) = 0 \,. \quad (1)$$

Here x_i is the offset of *i*-th particle, Ω_i, Q_i, ΔQ_{sc} are its revolution frequency, the tune and the direct space charge tune shift, Ω_0, Q_0 are the average revolution frequency and tune, $\bar{x}$ is the offset of beam center and ΔQ_c is the impedance-driven coherent tune shift. This equation was derived assuming that the beam oscillates as a rigid body. Consequently, the beam coherent motion is completely described by the dipole offset $\bar{x}$. This assumption is correct if all lattice frequencies $\Omega_i Q_i$ are identical. In this case, all particles respond identically to the coherent field, so that changes in amplitudes of different particles driven by the coherent field are equal. It yields $\delta x_i = \bar{x}$, resulting in the beam oscillating as a rigid body. However, a spread of the lattice frequencies generally makes the rigid-body model of Eq. (1) incorrect. Indeed, an individual response to the coherent field is determined by the separation of the individual lattice frequency from the coherent frequency, which varies from particle to particle. Since individual responses are not identical, the beam shape is not preserved with the dipole oscillations, so the rigid-body model with its Eq. (1) is not self-consistent and generally cannot be justified.

In 2001, M. Blaskiewicz showed a way to analyze the problem, avoiding the rigid beam assumption [2]. Within a one-dimensional model, he developed a rather complicated integral equation on the phase space density perturbation. He found two cases when his equation gives the same result as the rigid-beam approach. The first case was the Lorentz distribution of chromatic frequencies, and the second one was the water-bag distribution over the transverse actions. With some additional model simplifications, he plotted several stability diagrams for distributions close to Gaussian. The same problem of self-consistent beam stability analysis was recently considered by D. Pestrikov [3]. Considering a two-dimensional model, he came to a general integral equation and found it "too complicated even for a numerical solving." To proceed, he accepted a simplification of zero emittance for the second plane[1], came to the same integral equation as M. Blaskiewicz, and reproduced his Lorentz and water-bag results. For a Gaussian distribution, he plotted some additional stability diagrams, and realized that an octupole Landau anti-damping, which he found earlier for the rigid-beam model [4], disappears. Indeed, Landau anti-damping cannot exist at all if the distribution is close to Gaussian: this is a mere consequence of the second law of thermodynamics. A Hamiltonian system in thermal equilibrium is always stable. Appearance of Landau anti-damping in the rigid-beam model is an example of how wrong the results of this model can be. Rigid-beam

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[1] The simplification of zero emittance does not literally mean zero emittance in the other plane. Effectively it switches off the particle motion in the other plane and reduces the problem to a single dimension.

stability diagrams were presented in several papers [4-6], but since the model can lead to wrong results, the range of its applicability has to be quantitatively clarified.

MODEL JUSTIFICATION

As mentioned above, the rigid-beam model is exactly correct if all the lattice frequencies are identical, $\Omega_i\, Q_i = \Omega_0\, Q_0$. This case is simple, but not so interesting, since there is no Landau damping, and any impedance makes the beam unstable. Now let us assume some small lattice frequency spread, so small that the rigid-beam model would still be a good approximation. Specifically, this requires the rms spread of the lattice frequencies $\sigma(\Omega_i\, Q_i)$ to be small compared with the separation frequency, which is a difference of the coherent frequency from an average incoherent one:

$$\sigma(\Omega_i\, Q_i) << \Delta\Omega_{\rm sep} \equiv \Omega_0\left(\mathrm{Re}(\Delta Q_c) - \Delta\overline{Q}_{sc}\right) \; , \qquad (2)$$

where $\Delta\overline{Q}_{sc}$ is the average space charge tune shift. In this case, the rigid-beam model is still good, but due to tails of the distribution, there is some amount of the resonant particles resulting in the Landau damping. In this case its rate Λ_L is small, compared not only with the separation frequency $\Delta\Omega_{\rm sep}$, but also with the lattice frequency spread $\Lambda_L << \sigma(\Omega_i\, Q_i) << \Delta\Omega_{\rm sep}$. However, if the impedance-driven instability rate $\Omega_0\,\mathrm{Im}(\Delta Q_c)$ is also small, even this tiny amount of Landau damping can be sufficient for the beam stabilization. Thus, when the instability rate is much smaller than the separation frequency, or

$$\mathrm{Im}(\Delta Q_c) << \mathrm{Re}(\Delta Q_c) - \Delta\overline{Q}_{sc} \; , \qquad (3)$$

the required Landau damping is small. Consequently, the required lattice frequency spread is much smaller than the separation frequency. As a result, the frequency spread is not significant for a bulk of the beam, and the rigid beam model is a good approximation. In other words, when Eq. (3) is satisfied, the rigid-beam model is applicable for calculation of Landau damping required for beam stabilization.

The above consideration can also be presented in a somewhat different way. Eq. (2) determines that we may consider the beam as a core with identical lattice frequencies and a tiny tail whose frequency spread is high. At first approximation, the core oscillates as if there is no lattice frequency spread at all, so its oscillations are rigid. Since the tail is thin, its influence on the core oscillations is weak, and its motion in the field of the core can be considered as driven in a strong-weak approximation. For a majority of the tail particles these driven oscillations do not matter much since they are detuned from the core coherent motion. However, a small fraction of tail particles, which is resonant with the core, plays a significant role. The resonant particles of the tail absorb the core coherent energy, damping the core coherent motion as a result. This energy-based calculation of Landau damping leads to the same result as a formal solution of the dispersion equation [7].

In this paper, we limit ourselves to a case of thin tail, or small frequency spread approximation of Eq. (2), where the rigid-beam model is applicable. This allows us to calculate the Landau damping and the threshold parameters of the beam for a relatively small growth rate (3). Our primary interest is the threshold calculation. This is additionally simplified due to exponentially small phase space density of resonant particles, and consequently the Landau damping. That is why the threshold is mostly set by large value of the dimensionless frequency separation $\Delta\Omega_{\rm sep}/\sigma(\Omega_i\, Q_i)$, and the dependence of threshold on the coherent growth rate (impedance) is weak (logarithmical). For most practical cases, the expected accuracy of the Landau damping rate calculations is typically ~15-30% due to inaccuracy of the rigid beam model. However, the corresponding accuracy of the instability threshold is much higher due to its weak (logarithmic) dependence on the Landau damping rate. In practice, the far tails of the distributions are not well-measured, or highly reproducible. That is why even exact formulas can result in poor accuracy for the damping rate. On the contrary, the stability threshold is predicted much better. Note also that the condition of small impedance of Eq. (3) is typically well-satisfied for low and medium energy hadron machines. That justifies the application of the rigid-beam model for the threshold calculation.

DISPERSION EQUATION

After validity limits of the rigid-beam model are specified, a solution of Eq. (1) can be considered in more detail. Assuming $x_i(t) \propto \exp(-i\omega t)$ and $\omega \equiv \Omega_0(n + Q_0 + \nu)$, the dispersion relation for the eigenvalue ν follows [1]:

$$\begin{aligned}
&\varepsilon(\nu) \equiv 1 - \frac{1}{2}\int \frac{\left(\Delta Q_c(\omega) - \Delta Q_{sc}(a_x, a_y)\right) f_x a_x d\Gamma}{\Delta Q_l(a_x, a_y, \hat{p}) + \Delta Q_{sc}(a_x, a_y) - \nu - i0} = 0;\\
&d\Gamma \equiv a_x a_y da_x da_y d\hat{p};\\
&\Delta Q_l(a_x, a_y, \hat{p}) \equiv (\xi - n\eta)\hat{p} + \Delta Q_o(a_x, a_y);\; n = 0, \pm 1, \pm 2, ...\\
&\delta\nu = \omega/\Omega_0 - (n + Q_0)\,.\\
&f = f(a_x, a_y, \hat{p});\quad f_x \equiv \frac{\partial f}{\partial a_x};\quad \int f\, a_x a_y da_x da_y d\hat{p} = 1.
\end{aligned} \qquad (4)$$

Here, all the notations are rather conventional: a_x and a_y are two transverse amplitudes normalized by the beam rms sizes $x = a_x\sqrt{\varepsilon_x \beta_x}\cos\psi_x$ and similar for y; $\hat{p} = \Delta p/p$ is a relative momentum offset; $\Delta Q_l(a_x, a_y, \hat{p})$ is the total lattice-related tune shift; $\Delta Q_o(a_x, a_y)$ is its (octupole) nonlinear part; $\Delta Q_{sc}(a_x, a_y)$ is the direct space charge tune shift as a function of the amplitudes; and $\eta = 1/\gamma_t^2 - 1/\gamma^2$ is the slippage factor. The coherent shift $\Delta Q_c(\omega)$ describes the beam interaction with the wall. This interaction produces both the dipole and quadrupole forces, or, in other words, driving and detuning wakes [8].

Thus, the entire force acting on i-th particle can be expressed as $F_i = W\bar{x} + Dx_i$, with W as the conventional dipole, or driving wake function, and D as the quadrupole, or detuning wake function. For a continuous beam we can consider that only the driving wake leads to the coherent shift $\Delta Q_c(\omega) \propto W$, because the detuning wake simply shifts all frequencies by the same amount and therefore can be omitted.

A conventional method of analysis of the dispersion equation consists in drawing stability diagrams for various cases. According to what was discussed above, this procedure does not make much sense for the rigid-beam model if inequalities of Eqs. (2) and (3) are not fulfilled. Indeed, the stability diagram pretends to show a stability limit for a wide range of the coherent shifts $\Delta Q_c(\omega)$ while it is applicable only to the tails of the stability diagram. The most impressive example of how wrong it can be is the above-mentioned Landau antidamping, erroneously predicted by this model for negative octupole nonlinearity [4,6]. Thus, the stability diagram obtained within the rigid-beam model has to be zoomed in for the small area of Eq. (2) and disregarded as invalid for the rest of the complex plane. In that area, however, another significant step can be done: the rate of Landau damping can be calculated and expressed in terms of a regular integral of the distribution function f.

LANDAU DAMPING

When Eq. (2) is satisfied, the rate of Landau damping can be found from Eq. (4). Note that this dispersion equation formally defines the dielectric function $\varepsilon(\nu)$ for values of ν located on the real axis and the upper half-plane, $\mathrm{Im}(\nu) \geq 0$. To obtain it in the lower half-plain, where roots of the dispersion equation are located, the direct use of Eq. (4) is invalid; instead, a complex extension of analytical function $\varepsilon(\nu)$ has to be used. This can be done in the following way. First, let the eigenvalue ν be real, and solve the dispersion equation for the coherent shift as a function of the eigenvalue. Then the imaginary part of the coherent shift is equal to the Landau damping at the boundary of stability. The result is a straightforward expansion over a small parameter of the relative tune spread $\Delta Q_l / \Delta Q_{sep}$. For the real and imaginary parts of the eigenvalue, one can write:

$$\nu = \mathrm{Re}\Delta Q_c + \delta Q^{(1)} + \delta Q^{(2)},$$

$$\Lambda \equiv \mathrm{Im}\Delta Q_c = -\frac{\pi}{2}\langle \Delta Q_{sep}\rangle \int \Delta Q_{sep} f_x a_x\, \delta(\Delta Q_l + \Delta Q_{sc} - \nu) d\Gamma, \tag{5}$$

$$\Delta Q_{sep} \equiv \mathrm{Re}\Delta Q_c - \Delta Q_{sc}(a_x, a_y), \quad \langle \Delta Q_{sep}\rangle \equiv -2\left(\int \frac{f_x a_x\, d\Gamma}{\Delta Q_{sep}}\right)^{-1},$$

$$\delta Q^{(1)} = -\frac{\langle \Delta Q_{sep}\rangle}{2}\int \frac{\Delta Q_l f_x a_x\, d\Gamma}{\Delta Q_{sep}}.$$

$$\delta Q^{(2)} = -\frac{\langle \Delta Q_{sep}\rangle}{2}\int \frac{\Delta Q_l^2 f_x a_x\, d\Gamma}{\Delta Q_{sep}^2}.$$

Note that a sign of the damping rate Λ is always determined by a sign of the derivative of the distribution function $f_x = \partial f / \partial a_x$ for the resonance particles, similar to the classical Landau result for the plasma oscillations (no antidamping for monotonic distributions). Note also that corrections $\delta Q^{(1)}$ and $\delta Q^{(2)}$ to the real part of the eigenvalue play an important role because the distribution function changes fast, and even a small correction to the tune of the resonant particles significantly affects the damping rate Λ.

Let us first assume that the tune spread is purely chromatic. In this case the first-order correction is equal to zero, $\delta Q^{(1)} = 0$, and only the second-order correction remains, $\delta Q^{(2)}$. For the Gaussian momentum distribution, $f \propto \exp(-\hat{p}^2 / 2\sigma_p^2)$, and constant transverse density, $\Delta Q_{sc} = \mathrm{const}$, the second-order correction is determined by the following equation: $\delta Q^{(2)} = \sigma_{\nu p}^2 / \Delta Q_{sep}$, where $\sigma_{\nu p} \equiv |\xi - n\eta| \sigma_p$. That yields the damping rate:

$$\Lambda = \sqrt{\frac{\pi}{2}} \frac{\Delta Q_{sep}^2}{\sigma_{\nu p}} \exp\left(-\frac{\Delta Q_{sep}^2}{2\sigma_{\nu p}^2} - 1\right). \tag{6}$$

This result is e times smaller than a simple-minded result neglecting the second-order term $\delta Q^{(2)}$.

Another possibility to stabilize the beam is an introduction of octupole non-linearity. Contrary to the case of chromatic spread, the first-order correction to the eigenvalue $\delta Q^{(1)}$ is non-zero. For the Gaussian transverse distribution, accounting this first-order correction reduces the rate Λ by a constant factor ~2-3, similar to the role of the second-order term for the chromatic tune spread. For the octupole tune spread the second-order term makes only small correction to the damping rate and can be neglected.

As was pointed out above, the rigid-beam model is valid only if the frequency spread is small compared with the separation frequency, $\delta Q^{(1,2)} / \langle \Delta Q_{sep} \rangle << 1$. Accounting the tune corrections $\delta Q^{(1)}, \delta Q^{(2)}$ within the rigid-beam approximation assumes that inaccuracy of the model is smaller than these corrections. Answering this question is a subject of separate study. At the moment, we can only refer to a specific example of chromatic tune spread for a Gaussian beam, considered in Ref. [3] within a framework of one-dimensional self-consistent model, compared with the rigid-beam result. As it is clearly seen from a presented stability diagram, the discrepancy between the two results is rather small, ~ 10-20% in the area of rigid-beam validity. This suggests that accounting the eigenvalue corrections $\delta Q^{(1)}, \delta Q^{(2)}$ is within the model accuracy, and thus it is legitimate. Finally, it should be noted that although the corrections $\delta Q^{(1)}, \delta Q^{(2)}$ change the damping rate Λ by 2-3 times, their influence on the threshold space charge over the tune spread value is relatively small, since the Landau damping exponentially depends on beam parameters (like in Eq. (6)), and an error in the pre-exponential factor (~2-3) only slightly modifies the threshold.

THRESHOLD LINES

As it was stated above, the conventional stability diagrams, based on the rigid-beam assumption, are mostly invalid if the space charge is present. A small correct part of them lies typically so close to zero that it is hard to resolve details on the pictures usually presented in the literature (Ref. [4-6]). Since these diagrams are mostly either misleading or useless, we do not draw them here and present the stability threshold in a different way. Indeed, Eq. (6) shows that the stability condition depends on the two dimensionless parameters. The first parameter determines to what extent the coherent and incoherent frequencies are separated; obviously, it is defined by the ratio of the separation frequency over the lattice frequency spread. The second parameter shows how strong is the instability to be suppressed by the Landau damping; it can be described by the coherent growth rate $\Omega_0 \,\mathrm{Im}\Delta Q_c$ in units of the separation frequency. A dependence of the threshold dimensionless separation over dimensionless coherent growth can be called the threshold line. In this section we present it for the case of round Gaussian beam. The problem is solved for the cases of the pure chromatic tune spread with $\Delta Q_l = (\xi - n\eta)\hat{p} \equiv \sigma_{vp}\hat{p}/\sigma_p$, and the tune spread introduced only by octupole non-linearities equal for both planes, $\Delta Q_l = \Delta Q_o = \sigma_{vo}(a_x^2 + a_y^2)/2$. The results are presented in Figures. 1, 2.

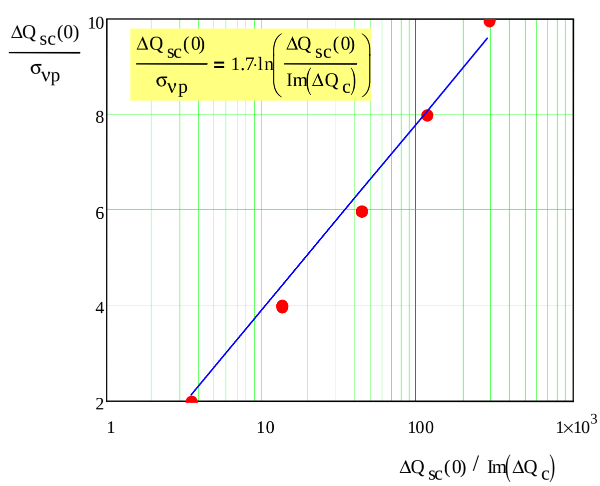

Figure 1: Threshold line for the chromatic tune spread. The dimensionless maximal space charge tune shift $\Delta Q_{sc}(0)/\sigma_{vp}$ is plotted versus dimensionless growth time $\Delta Q_{sc}(0)/\mathrm{Im}\Delta Q_c$. The dots are numerical results, and the line is a fit with the formula highlighted in yellow.

To compute the space charge tune shift for a round Gaussian beam, $\sigma_x = \sigma_y = \sqrt{\varepsilon\beta_x}$, we used a conventional formula:

$$\Delta Q_{sc}(a_x,a_y) = -\Delta Q_{sc}(0)\int_0^1 \frac{\left[I_0\left(\frac{a_x^2 z}{4}\right) - I_1\left(\frac{a_x^2 z}{4}\right)\right] I_0\left(\frac{a_y^2 z}{4}\right)}{\exp\left(z\left(a_x^2+a_y^2\right)/4\right)}\, dz, \quad (7)$$

where $\Delta Q_{sc}(0) = r_p \lambda C/(4\pi\beta^2\gamma^3\varepsilon)$ is the maximal space charge tune shift, and λ is the linear density. For numerical calculations, we used the following fit,

$$\Delta Q_{sc}(a_x,a_y) = -\Delta Q_{sc}(0)\frac{200+30a_y-10a_x-7a_x a_y}{200+40a_x^2-10a_x+30a_y^2+30a_y}, \quad (8)$$

which is accurate within few percent for $a_x, a_y \le 6$.

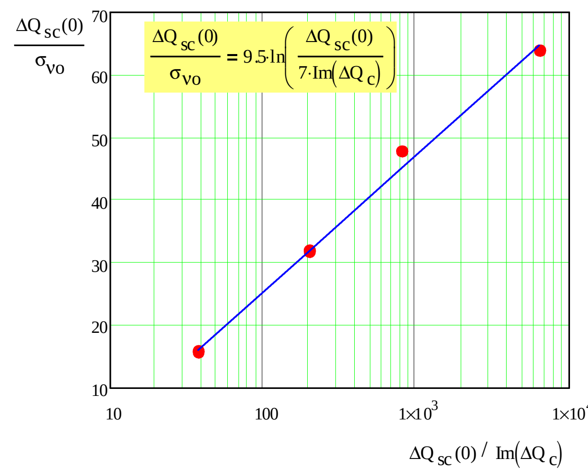

Figure 2: Threshold line for the octupole tune spread.

There is a significant difference between the two plots: for the octupoles, the stabilizing tune spread is 3-4 times smaller than required for the chromatic tune spread. The reason is that the octupole-driven tune shift goes quadratically with amplitudes, while the chromatic tune shift is a linear function of the momentum offset.

SUMMARY

The applicability of the rigid-beam model is considered for the case when the space charge plays significant role in beam dynamics. The results prove that the stability diagrams obtained with this model are not valid for most of the complex plane of the coherent shift. However, the small area where it is valid typically covers the entire area of practical interest. Based on the rigid-beam model, rather simple formulas for the Landau damping were calculated. These formulas are used for the calculation of the threshold space charge tune shift versus coherent growth time. Convenient analytical fits for the threshold lines are presented.

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