COUPLING AND ITS EFFECTS ON BEAM DYNAMICS*

A. Burov and V. Lebedev†

FNAL, Batavia, IL 60510, U.S.A.

Abstract

Coupling between different degrees of freedom complicates analysis of beam dynamics in a ring. Nevertheless appropriate choice of dynamic variables often allows reducing a problem to uncoupled case. Effects of coupling on the beam instabilities and their damping are considered using the extended Mais-Ripken parameterization for X-Y coupled motion.

PARAMETERIZATION OF COUPLED MOTION

We will use the extended Mais-Ripken parameterization to describe the coupling between two degrees of freedom. This introduction presents definitions and major results required for further consideration. Details and proofs can be found in Ref. [1].

The eigenvectors are parameterized in the following form:

$$\mathbf{v}_1=\begin{bmatrix}\sqrt{\beta_{1x}}\\ -\dfrac{i(1-u)+\alpha_{1x}}{\sqrt{\beta_{1x}}}\\ \sqrt{\beta_{1y}}e^{i\nu_1}\\ -\dfrac{iu+\alpha_{1y}}{\sqrt{\beta_{1y}}}e^{i\nu_1}\end{bmatrix},\quad \mathbf{v}_2=\begin{bmatrix}\sqrt{\beta_{2x}}e^{i\nu_2}\\ -\dfrac{iu+\alpha_{2x}}{\sqrt{\beta_{2x}}}e^{i\nu_2}\\ \sqrt{\beta_{2y}}\\ -\dfrac{i(1-u)+\alpha_{2y}}{\sqrt{\beta_{2y}}}\end{bmatrix}, \tag{1}$$

where β- and α-functions are similar to the corresponding values of uncoupled case, u characterizes the strength of coupling, and ν's characterize the phase differences between horizontal and vertical motion. Symplecticity requires that only 8 of 11 functions are independent (see [1] for details). The eigenvectors are normalized so that

$$\begin{cases}\mathbf{v}_n^{\ +}\mathbf{U}\mathbf{v}_n=-2i, & n=1,2\\ \mathbf{v}_n^{\ +}\mathbf{U}\mathbf{v}_m=0, & n\neq m\\ \mathbf{v}_n\mathbf{U}\mathbf{v}_m=0, & any\ n,m\end{cases} \tag{2}$$

where $\mathbf{U}$ is the unit symplectic matrix,

$$\mathbf{U}=\begin{bmatrix}\mathbf{U}_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{U}_2\end{bmatrix},\quad \mathbf{U}_2=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}. \tag{3}$$

A single particle motion can be presented in the following form:

$$\mathbf{x}(s)=\mathrm{Re}\left(\sqrt{\varepsilon_1}\mathbf{v}_1(s)e^{-i(\psi_1+\mu_1(s))}+\sqrt{\varepsilon_2}\mathbf{v}_2(s)e^{-i(\psi_2+\mu_2(s))}\right). \tag{4}$$

where vector $\mathbf{x}(s)=\left[x(s),\theta_x(s),y(s),\theta_y(s)\right]^T$ describes particle coordinates, μ's are the betatron phase advances, s is the path length, and ε_1 and ε_2 are the single particle emittances (generalized Courant-Snyder invariants). If longitudinal magnetic field is equal to zero, θ_x and θ_y can be considered as particle angles, otherwise they have to be considered as generalized momenta. Dependence on s was explicitly shown in Eq. (4). We omit this notation below.

Gaussian distribution function can be written in the following form

$$f(\mathbf{x})=\frac{1}{4\pi^2\varepsilon_1\varepsilon_2}\exp\left(-\frac{\mathbf{x}^T\mathbf{\Xi}\mathbf{x}}{2}\right), \tag{5}$$

where

$$\mathbf{\Xi}=\mathbf{U}\mathbf{V}\mathbf{\Xi}'\mathbf{V}^T\mathbf{U}^T, \tag{6}$$

$$\mathbf{V}=\left[\mathrm{Re}(\mathbf{v}_1),-\mathrm{Im}(\mathbf{v}_1),\mathrm{Re}(\mathbf{v}_2),-\mathrm{Im}(\mathbf{v}_2)\right], \tag{7}$$

$$\mathbf{\Xi}'=\begin{bmatrix}\boldsymbol{\varepsilon}_1^{-1} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\varepsilon}_2^{-1}\end{bmatrix},\quad \boldsymbol{\varepsilon}_n=\begin{bmatrix}\varepsilon_n & 0\\ 0 & \varepsilon_n\end{bmatrix},\quad n=1,2, \tag{8}$$

and ε_1 and ε_2 are emittances of modes 1 and 2. Note that the total 4D emittance of the beam is equal to the product of mode emittances, $\varepsilon_{4D}=\varepsilon_1\varepsilon_2$, where we omit the factor $\pi^2/2$ correcting for the volume of 4D ellipsoid. The inversion of $\mathbf{\Xi}$ matrix results in the matrix of second moments $\langle x_i x_j\rangle\equiv\mathbf{\Sigma}=\mathbf{\Xi}^{-1}$. In particular the rms beam sizes are:

$$\begin{aligned}\langle x^2\rangle&\equiv\sigma_x^{\ 2}=\varepsilon_1\beta_{1x}+\varepsilon_2\beta_{2x},\\ \langle y^2\rangle&\equiv\sigma_y^{\ 2}=\varepsilon_1\beta_{1y}+\varepsilon_2\beta_{2y},\\ \langle xy\rangle&\equiv\sigma_{xy}^{\ 2}=\varepsilon_1\sqrt{\beta_{1x}\beta_{1y}}\cos\nu_1+\varepsilon_2\sqrt{\beta_{2x}\beta_{2y}}\cos\nu_2.\end{aligned} \tag{9}$$

PERTURBATION THEORY

For unperturbed motion the eigenvectors and eigenvalues are related to the transfer matrix $\mathbf{M}$ as following:

$$\mathbf{M}\mathbf{v}_j=\lambda_j\mathbf{v}_j,\quad j=1,...4. \tag{10}$$

Then for the perturbed motion one can write:

$$(\mathbf{M}+\Delta\mathbf{M})\tilde{\mathbf{v}}_j=(\lambda_j+\Delta\lambda_j)\tilde{\mathbf{v}}_j, \tag{11}$$

where the perturbed eigenvectors are presented as a sum of unperturbed ones,

$$\tilde{\mathbf{v}}_j=\mathbf{v}_j+\sum_{i=1}^{4}\varepsilon_{ij}\mathbf{v}_i,\quad \varepsilon_{ij}<<1, \tag{12}$$

and without limitation of generality one can consider that $\varepsilon_{ii}=0$. Substituting Eq. (12) into Eq. (11), linearizing the resulting equation, and using Eq. (10), one obtains:

$$\sum_{i=1}^{4}(\lambda_i-\lambda_j)\varepsilon_{ij}\mathbf{v}_i=(\Delta\lambda_j\mathbf{I}-\Delta\mathbf{M})\mathbf{v}_j, \tag{13}$$

where $\mathbf{I}$ is the identity matrix. In the case of stable motion

*Work supported by the US DoE contract No. DE-AC02-07CH11359

† val@fnal.gov

the eigenvalues and eigenvectors represent two complex conjugate pairs. Taking this into account, $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4\end{bmatrix} \to \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_1^* & \mathbf{v}_2 & \mathbf{v}_2^*\end{bmatrix}$, and introducing complex matrix $\mathbf{V}_c = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_1^* & \mathbf{v}_2 & \mathbf{v}_2^*\end{bmatrix}$ one can rewrite Eq. (13) in the form of two matrix equations:

$$\mathbf{V}_c \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 - \lambda_1^* & 0 & 0 \\ 0 & 0 & \lambda_1 - \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_1 - \lambda_2^* \end{bmatrix} \begin{bmatrix} \Delta\lambda_1 \\ \varepsilon_{21} \\ \varepsilon_{31} \\ \varepsilon_{41} \end{bmatrix} = \mathbf{\Delta M v}_1 ,$$
$$\mathbf{V}_c \begin{bmatrix} \lambda_2 - \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 - \lambda_1^* & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 - \lambda_2^* \end{bmatrix} \begin{bmatrix} \varepsilon_{12} \\ \varepsilon_{22} \\ \Delta\lambda_2 \\ \varepsilon_{42} \end{bmatrix} = \mathbf{\Delta M v}_2 . \tag{14}$$

Matrix V_c consists of symplectic vectors and its inverse is equal to:

$$\mathbf{V}_c^{-1} = -\frac{1}{2i}\mathbf{U}\mathbf{V}_c^T\mathbf{U} \ . \tag{15}$$

One can verify it by utilizing the eigenvector normalization of Eqs. (2). Inversion of Eq. (14) with help of Eq. (15) finally results in:

$$\begin{bmatrix} \Delta\lambda_1 \\ \varepsilon_{21} \\ \varepsilon_{31} \\ \varepsilon_{41} \end{bmatrix} = -\frac{1}{2i} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 - \lambda_1^* & 0 & 0 \\ 0 & 0 & \lambda_1 - \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_1 - \lambda_2^* \end{bmatrix}^{-1} \mathbf{U}\mathbf{V}_c^T\mathbf{U}\,\mathbf{\Delta M}\,\mathbf{v}_1$$
$$\begin{bmatrix} \varepsilon_{12} \\ \varepsilon_{22} \\ \Delta\lambda_2 \\ \varepsilon_{42} \end{bmatrix} = -\frac{1}{2i} \begin{bmatrix} \lambda_2 - \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 - \lambda_1^* & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 - \lambda_2^* \end{bmatrix}^{-1} \mathbf{U}\mathbf{V}_c^T\mathbf{U}\,\mathbf{\Delta M}\,\mathbf{v}_2 \tag{16}$$

Multiplication of Eqs. (16) by $\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}$, correspondingly, results in corrections for the eigenvalues:

$$\Delta\lambda_1 = -\frac{1}{2i}\ \mathbf{v}_1^{+}\mathbf{U}\,\mathbf{\Delta M}\,\mathbf{v}_1 ,$$
$$\Delta\lambda_2 = -\frac{1}{2i}\ \mathbf{v}_2^{+}\mathbf{U}\,\mathbf{\Delta M}\,\mathbf{v}_2 . \tag{17}$$

TUNE SHIFTS

Let us find the tune shifts due to a local focusing perturbation. In the general case the perturbation of the Hamiltonian is proportional to $\Phi_x x^2 + 2\Phi_s xy + \Phi_y y^2$. That results in the transfer matrix of the perturbation:

$$\mathbf{M}_q = \mathbf{I} + \mathbf{\Delta M}_q , \quad \mathbf{\Delta M}_q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\Phi_x & 0 & -\Phi_s & 0 \\ 0 & 0 & 0 & 0 \\ -\Phi_s & 0 & -\Phi_y & 0 \end{bmatrix} . \tag{18}$$

Then the perturbation to the transfer matrix (see Eq. 11) is $\mathbf{\Delta M} = \mathbf{\Delta M}_q\mathbf{M}$. Substituting it to Eqs. (17) and taking into account the relationship between the eigenvalue corrections and the tune shifts, $\Delta Q_n = i/(4\pi)\left(\Delta\lambda_n / \lambda_n\right)$, one obtains [2]*:

$$\Delta Q_1 = -\frac{1}{4\pi}\mathbf{v}_1^{+}\mathbf{U}\,\mathbf{\Delta M}_q\,\mathbf{v}_1 ,$$
$$\Delta Q_2 = -\frac{1}{4\pi}\mathbf{v}_2^{+}\mathbf{U}\,\mathbf{\Delta M}_q\,\mathbf{v}_2 . \tag{19}$$

Performing multiplications using the eigenvectors of Eq. (1) one finally obtains:

$$\Delta Q_1 = \frac{1}{4\pi}\left(\Phi_x\beta_{1x} + 2\Phi_s\sqrt{\beta_{1x}\beta_{1y}}\cos\nu_1 + \Phi_y\beta_{1y}\right),$$
$$\Delta Q_2 = \frac{1}{4\pi}\left(\Phi_x\beta_{2x} + 2\Phi_s\sqrt{\beta_{2x}\beta_{2y}}\cos\nu_2 + \Phi_y\beta_{2y}\right). \tag{20}$$

One can see that in the case of uncoupled motion, $\beta_{1y} = \beta_{2x} = 0$, the tune shifts coincide with the well-known expression for the tune shift of uncoupled motion.

TRANSVERSE INSTABILITIES IN X-Y COUPLED CASE

First, let us find how amplitudes of betatron motion, a_1 and a_2, are changed due to a single local kick. Using Eq. (4) one can express the vector of particle coordinates through the amplitudes

$$\mathbf{x} = \frac{1}{2}\left(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + C.C.\right) \quad , \tag{21}$$

where $C.C.$ denotes the complex conjugate. If kicks in the horizontal and vertical planes are equal to $\delta\theta_x$ and $\delta\theta_y$ a change of particle vector is:

$$\mathbf{x}' = \mathbf{x} + \mathbf{\delta x} \quad , \tag{22}$$

with $\mathbf{\delta x} = \begin{bmatrix}0 & \delta\theta_x & 0 & \delta\theta_x\end{bmatrix}^T$. To find changes of the amplitudes, we substitute the particle vectors in the form of Eq. (21) to Eq. (22). That results in

$$\mathbf{\delta x} = \frac{1}{2}\left(\delta a_1\mathbf{v}_1 + \delta a_2\mathbf{v}_2 + C.C.\right) \quad , \tag{23}$$

Multiplying it by $\mathbf{v}_1^{+}\mathbf{U}$ or $\mathbf{v}_2^{+}\mathbf{U}$ and using orthogonality conditions of Eq. (2) one obtains:

$$\delta a_1 = i\mathbf{v}_1^{+}\,\mathbf{U}\,\mathbf{\delta x} = \sqrt{\beta_{1x}}\delta\theta_x + e^{-i\nu_1}\sqrt{\beta_{1y}}\delta\theta_y ,$$
$$\delta a_2 = i\mathbf{v}_2^{+}\,\mathbf{U}\,\mathbf{\delta x} = e^{-i\nu_2}\sqrt{\beta_{2x}}\delta\theta_x + \sqrt{\beta_{2y}}\delta\theta_y . \tag{24}$$

To compute changes of betatron amplitudes due to beam interaction with vacuum chamber, we express the transverse kicks through the transverse impedances per unit length,

$$\begin{bmatrix}\delta\theta_x \\ \delta\theta_y\end{bmatrix} = -i\frac{eI_b}{\beta^2\gamma mc^2}\begin{bmatrix}Z_x x \\ Z_y y\end{bmatrix}\delta s \ . \tag{25}$$

Substituting the above equations in Eqs. (24) and expressing the displacements through the amplitudes using Eq. (21), one obtains:

* Note that there is close resemblance between the tune shift of Eq. (19) and the energy shift of perturbed level in the quantum mechanics $\Delta E \propto \langle\psi \mid \Delta U \mid \psi\rangle$. Appearance of matrix **U** in Eq. (19) is related to the orthogonality conditions of Eq. (2).

$$\begin{bmatrix}\delta a_1\\ \delta a_2\end{bmatrix} = \mathbf{\Delta M}_Z \begin{bmatrix} a_1 \\ a_2 \end{bmatrix},$$

$$\mathbf{\Delta M}_Z = -\frac{ieI_b}{\beta^2\gamma mc^2}\begin{bmatrix}\beta_{1x}Z_x+\beta_{1y}Z_y & (\beta Z)_C \\ \overline{(\beta Z)_C} & \beta_{1x}Z_x+\beta_{1y}Z_y\end{bmatrix}, \quad (26)$$

$$(\beta Z)_C = \sqrt{\beta_{1x}\beta_{2x}}Z_x e^{i\nu_2} + \sqrt{\beta_{1y}\beta_{2y}}Z_y e^{-i\nu_1} .$$

where $\overline{(\beta Z)_C}$ denotes the complex conjugate of $(\beta Z)_C$, I_b is the beam current, β and γ are the relativistic factors, and e and m are the particle charge and mass. In the above equation we took into account that the phase averaging nullifies terms proportional to the complex conjugated values of a_1 and a_2 if the instability growth rates are much smaller than the distance to the nearest half integer or difference coupling resonance:

$$\lambda T << \left|\mu_1+\mu_2-2\pi n\right|, \left|\mu_{1,2}-\pi n\right| . \quad (27)$$

If the ring impedance is dominating the beam dynamics and other phenomena (Landau damping, space charge, etc.) can be neglected the solution of dispersion equation,

$$\left|\mathbf{M}(\mathbf{I}+\mathbf{\Delta M}_Z)-\lambda\mathbf{I}\right| = 0 \quad , \quad (28)$$

yields the eigenvalues and, consequently, the instability growth rates.

Nevertheless if in addition to conditions of Eq. (27) the instability growth rates are smaller than the tune separation,

$$\lambda T << \left|\mu_1-\mu_2-2\pi n\right| \quad , \quad (29)$$

the phase averaging nullifies the off-diagonal elements in matrix $\mathbf{\Delta M}_Z$ and the modes are decoupled. This reduces the problem to the much better known problem of uncoupled instability resulting in that the entire single plane theory (both for bunched and unbunched beam) works with the following substitutions [3]:

$$\begin{aligned} &\beta_x Z_x \to (\beta_{1x}Z_x+\beta_{1y}Z_y),\ \nu_x \to \nu_1 \ \text{ - for mode 1 },\\ &\beta_y Z_y \to (\beta_{2x}Z_x+\beta_{2y}Z_y),\ \nu_y \to \nu_2 \ \text{ - for mode 2 }. \end{aligned} \quad (30)$$

As example we consider the dispersion equation for continuous beam with space charge of Ref. [4]. For the mode 1 it is:

$$1-\frac{1}{2}\int\frac{\left(\Delta Q_{c1}-\Delta Q_{ic1}(a_1,a_2)\right)\dfrac{df}{da_1}a_1^{\,2}a_2\,da_1 da_1 d\hat{p}}{\xi_{n1}\hat{p}+\Delta Q_{lat1}(a_1,a_2)+\Delta Q_{ic1}(a_1,a_2)-\dfrac{\delta\omega_n}{\Omega_0}-i0} = 0 . \quad (31)$$

Comparing to Ref. [4] one can see that Eq. (31) was obtained by replacement of amplitudes a_x and a_y by a_1 and a_2. However one has to note that there are other details that need to be taken into account: (1) the chromaticity, ξ_{n1}, is the mode chromaticity and has to be computed with coupling taken into account; (2) the coherent tune shift, ΔQ_{c1}, has to be computed with Eq. (30); (3) the incoherent tune shift, ΔQ_{ic1}, and the tune shift due to lattice nonlinearity ΔQ_{lat1} have to be computed using Eq. (20) where additionally one needs to take into account that the beam field has both normal and skew quadrupole fields due to beam rotation described by Eq. (9) or by its extension if the dispersion contributions to the beam sizes are not negligible.

Note also that if the growth rate is much larger than the fractional part of tune separation, $\lambda T >> \left|\{\mu_1-\mu_2\}\right|$, and impedances are directed along x- and y-axes so that $\Delta\theta_x \propto Z_x x$ and $\Delta\theta_y \propto Z_y x$, the coupling can be neglected and growth rates can be computed for uncoupled beta-functions.[†]

EFFECT OF COUPLING ON TRANSVERSE DAMPER OPERATION

Digital bunch-by-bunch horizontal and vertical Tevatron dampers have been used to prevent development of transverse instabilities [5]. In the process of their commissioning and operation we found out that if coupling becomes too large a damper can unstabilize the motion in the orthogonal plane. That forced us to consider the damper operation in all details including effect of coupling considered below.

Pickups and kickers of both dampers are in the same straight line with no focusing elements in between. Figure 1 depicts the schematic for one of two dampers. In further consideration we will consider the horizontal damper only.

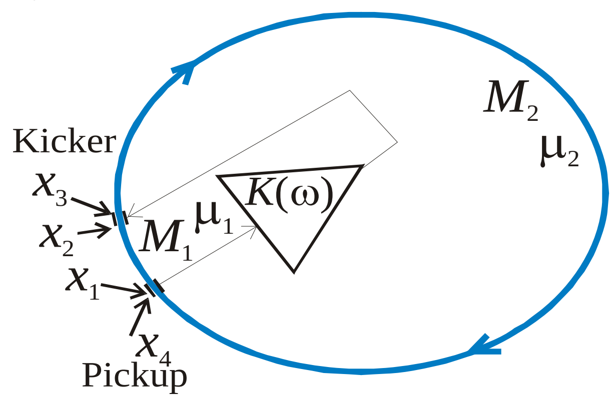

Figure 1: Transverse damper schematic

To suppress the effect of orbit offset on the damper operation, the kicker voltage is proportional to the beam position difference for two consecutive turns (digital notch filter). The kick is applied with additional one turn delay because of too large delay in electronics. It also results in appropriate betatron phase advance between pickup and kicker. The following sequence of equations describes the system:

$$\begin{aligned} \mathbf{x}_{2k} &= \mathbf{M}_1\mathbf{x}_{1k} ,\\ \mathbf{x}_{3k} &= \mathbf{x}_{2k} + \mathbf{G}(\mathbf{x}_{1k-1}-\mathbf{x}_{1k-2}),\\ \mathbf{x}_{1k+1} &= \mathbf{x}_{4k} = \mathbf{M}_2\mathbf{x}_{3k} , \end{aligned} \quad (32)$$

where the first index numerates the position along the ring (see Figure 1), the second index numerates the turn number and the kick matrix is determined as

[†] It is similar to the degenerate case in the quantum mechanics where perturbations of energy levels have to be calculated in the basis (linear combination of degenerated modes) which diagonalizes the Hamiltonian.

$$\begin{bmatrix} 0 \\ \delta\theta_x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta_x \\ y \\ \theta_y \end{bmatrix} \equiv \mathbf{Gx} \qquad (33)$$

Combining Eqs. (32) one obtains:

$$\mathbf{x}_{1k+1} = \mathbf{M}_2\left(\mathbf{M}_1\mathbf{x}_{1k} + \mathbf{G}\left(\mathbf{x}_{1k-1} - \mathbf{x}_{1k-2}\right)\right). \qquad (34)$$

Rewriting this equation through the eigenvectors and eigenvalues results in:

$$\lambda\mathbf{v} = \left(\mathbf{M} + \mathbf{M}_2\mathbf{G}\left(\lambda^{-1} - \lambda^{-2}\right)\right)\mathbf{v}, \qquad (35)$$

where we took into account that $\mathbf{M} = \mathbf{M}_2\mathbf{M}_1$. In the linear approximation the unperturbed eigen-values, $\lambda_n = e^{-i\mu_n}$, can be used in the perturbation term,

$$\Delta\mathbf{M} = \mathbf{M}_2\mathbf{G}\left(\lambda^{-1} - \lambda^{-2}\right) \quad . \qquad (36)$$

That results the following corrections for the eigen-vectors

$$\begin{aligned} \Delta\lambda_1 &= -e^{i\mu_1}\frac{1-e^{i\mu_1}}{2i}\mathbf{v}_1^{\,+}\mathbf{U}\,\mathbf{M}_2\mathbf{G}\,\mathbf{v}_1 \ , \\ \Delta\lambda_2 &= -e^{i\mu_2}\frac{1-e^{i\mu_2}}{2i}\mathbf{v}_2^{\,+}\mathbf{U}\,\mathbf{M}_2\mathbf{G}\,\mathbf{v}_2 \ . \end{aligned} \qquad (37)$$

The equations for vertical damper coincide with Eq. (37) but one needs to redefine the kick matrix of Eq. (33) so that the gain is moved from G_{21} to G_{43}.

Substituting numerical values of Tevatron lattice proved that a single plane damper can introduce instability in other plane if coupling is sufficiently strong. Simulations also showed that the major reasons of such behavior were significantly larger beta-function in the plane orthogonal to the damping plane (uncoupled optics) and the additional one turn delay which doubles the effect of coupling.

INSTABILITIES IN X-L COUPLED CASE

While strong coupling between longitudinal and transverse planes is not frequently encountered, it can be important if the longitudinal impedance is large at places with large dispersion. The beam interaction with axially symmetric high order modes of RF cavities located at non-zero dispersion can present an example of such interaction. Below we consider how such coupling could affect the longitudinal and transverse instabilities.

There is no difference in description of *x-y* coupling considered in Section 1 and *x-l* coupling. In this section we will consider that *y*-axis is directed along the beam direction which leaves all formulas in Section 1 intact. It also implies that $\Delta p_{\parallel}/p \equiv \Delta p_y/p$.

To proceed further we need to take into account the difference in the definitions of transverse, $Z_{\perp}$, and longitudinal, Z_L, impedances per unit length. By definition the longitudinal force acting on a particle is:

$$F_{\parallel\omega} = eI_\omega Z_{\parallel}(\omega) \quad . \qquad (38)$$

To express it through the particle displacement we take into account the relationship between Fourier harmonics of beam current perturbation and particle displacement:

$$I_\omega = iI_b k y_\omega \ , \quad k = \frac{\omega}{\beta c} \quad . \qquad (39)$$

That results in:

$$F_{\parallel\omega} = ieI_b k Z_{\parallel}(\omega) y_\omega \quad . \qquad (40)$$

In its turn the definition of the transverse impedance results in the force acting on a particle equal to:

$$F_{\perp\omega} = ieI_b Z_{\perp}(\omega) x_\omega \quad . \qquad (41)$$

Combining results of Section 4 with Eqs. (40) and (41) one finds that the following substitutions

$$\begin{aligned} &\beta_x Z_x \to \left(Z_x\beta_{1x} + kZ_{\parallel}\beta_{1s}\right), \quad \nu_x \to \nu_1 \ , \\ &k\beta_s Z_{\parallel} \to \left(Z_x\beta_{2x} + kZ_{\parallel}\beta_{2s}\right), \nu_s \to \nu_2 \ , \end{aligned} \qquad k = \frac{\omega}{v_0}, \qquad (42)$$

reduce the problem to a single dimensional problem similar to the *x-y* coupling considered in Section 4. However, in spite of obvious similarity presented above, there is significant difference in dynamics of longitudinal and transverse degrees of freedom. It is related to the large difference in tunes and higher relative non-linearity for longitudinal motion. Therefore one needs to be cautious applying results of *x-y* coupling to the *x-l* coupling.

CONCLUSIONS

In the case of coupling between two (or more) degrees of freedom an appropriate choice of dynamic variables allows to reduce a problem of beam stability to the case of a single degree of freedom. Symplecticity of the particle motion greatly simplifies analytical calculations and usage of the extended Mais-Ripken parameterization yields clear and physically meaningful results.

REFERENCES

[1] V. Lebedev, S. Bogacz, "Betatron motion with coupling of horizontal and vertical degrees of freedom", JLab-TN-00-022, http://www1.jlab.org/Ul/Publications/documents/JLAB-ACC-00-22.pdf, Jefferson Lab, 2000.

[2] A. Burov, "Two-beam instability in electron cooling", Phys Rev ST-AB, **9**, 120101 (2006).

[3] A. Burov, V. Lebedev, "Coherent motion with linear coupling", Phys Rev ST-AB, **10**, 044402 (2007).

[4] A. Burov, V. Lebedev, "Effect of Space Charge on Instabilities", This conference.

[5] C.Y. Tan, J. Steimel, "The Tevatron transverse damper", Proceedings of PAC-2003, p. 3074, 2003.