EQUIPARTITION, REALITY OR SWINDLE ?

Jean-Michel Lagniel, GANIL, Caen, France

Abstract

By way of introduction to a general discussion on space-charge induced energy equipartition (EQP), the following questions will be tackled : Where is the swindle ? Why the formula presently used to define EQP is wrong ? Why energy exchanges can occur although the EQP rule is respected ? Why safe tunings can be found although the EQP rule is not respected ? Why some linac designers nevertheless like to use the EQP rule ?

THE EQUIPARTITION THEOREM

The EQP theorem, also known as the "equipartition of energy principle", is a fundamental law in classical statistical mechanics. It states that the total energy of a system in thermal equilibrium is shared equally amongst all its energetically accessible independent degrees of freedom. In another way of saying that, the systems relevant of the classical statistical mechanics must distribute their available energy evenly amongst their independent accessible modes of motion when they are reaching a steady state.

For example, for an ideal mono-atomic gas with N "particles" confined in a box (3 translational degrees of freedom only, no rotational and vibrational degrees of freedom), it means that the average kinetic energies in every one of the 3N translational degrees of freedom shall be equal when the system will be in equilibrium.

The EQP theorem is here easily understandable looking to the microscopic level where the energy transfer induced by the collisions between the particles has an equal probability to be done towards the different degrees of freedom.

We must point out here that for this example, the EQP theorem concerns the 3N kinetic energies averaged over time of each one of the N particles

$$\frac{1}{T}\int_0^T v_{x1}^2\,dt = \frac{1}{T}\int_0^T v_{y1}^2\,dt = \frac{1}{T}\int_0^T v_{z1}^2\,dt = \ldots$$
$$= \frac{1}{T}\int_0^T v_{xi}^2\,dt = \frac{1}{T}\int_0^T v_{yi}^2\,dt = \frac{1}{T}\int_0^T v_{zi}^2\,dt = \ldots$$
$$= \frac{1}{T}\int_0^T v_{xN}^2\,dt = \frac{1}{T}\int_0^T v_{yN}^2\,dt = \frac{1}{T}\int_0^T v_{zN}^2\,dt$$
$$\text{when } T \to \infty \qquad (1)$$

For large N systems, (1) leads to equal mean kinetic energies in the x, y and z translational degrees of freedom, the averaging being done over the N particles at a given time

$$\frac{1}{N}\sum_{i=1}^{N} v_{xi}^2 = \frac{1}{N}\sum_{i=1}^{N} v_{yi}^2 = \frac{1}{N}\sum_{i=1}^{N} v_{zi}^2$$

or, written in a simplest form

$$\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle \qquad (2)$$

It is important to understand that the equality (2) which describes the macroscopic system behaviour is a consequence of the equality (1) which describes the microscopic behaviour of the systems when the EQP theorem can be applied.

EQP Theorem Validity Limit

The law of equipartition holds only for ergodic systems in thermal equilibrium, the ergodic hypothesis being considered as the basis of the statistical physics and an attempt to provide a bridge between dynamics and statistics. It basically asserts that the state ("trajectory") of an ergodic Hamiltonian system with n degrees of freedom, represented by a point in the 2n-phase space (q_1, ..., q_n, p_1, ..., p_n with q_i and p_i the generalized positions and momenta respectively) will pass equally often on every point of the constant-energy surface in this 2n-phase space during its long term evolution.

The ergodic hypothesis is still one of the most fascinating problems of physics and mathematics, subject of numerous discussions and publications (e.g. [1]) and by far out of the scope of this paper. In order to stick to the question of EQP applicability to our linac beams we will only recall that

A system is ergodic when the energy surface cannot be divided into finite regions such that, if the initial point in phase-space is located in one such region, the system trajectory remains entirely within that region (John von Neumann, 1932).

One of the very best descriptions of the Hamiltonian systems behaviour in this context is given in [2]. This paper shows how the complexity of the phase-space trajectories evolves with the nonlinearity level (weak / strong nonlinearity) and with the perturbing forces strength (K) which finally governs the global behaviour of the nonlinear systems :

- Complete integrability without perturbation (K = 0). The particles trajectories in phase space are ordered, their motions are quasiperiodic. The phase space trajectories of the resonant particles are represented by fix points and the non-resonant trajectories by continuous lines.
- KAM integrability for a weak perturbation (K → 0, weakly non-integrable Hamiltonian system). In this first level of disorder in phase space, most of the non-resonant trajectories are only slightly deformed but remains continuous ; then form a "KAM impassable barrier" which limits the accessible domain for the other particles. In the same time, the separatrix associated to the resonances are destroyed, narrow chaotic layers appears (Arnold and Avez, 1968).
- Complete chaos reached when the perturbation is increased (K → ∞). The particle motions are chaotic everywhere in phase space. The dynamics in such conditions becomes so erratic that the notion of phase space trajectory loses its meaning.

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The nonlinear Hamiltonian systems become ergodic only when the perturbations are strong enough to lead to the destruction of the KAM barriers nearly everywhere in phase space. This is the condition to have particle behaviours following equation (1), this is the condition which authorizes the application of the EQP theorem.

EQP AND LINAC BEAM DYNAMICS

The early beginning of "Equipartitionism" in the high-intensity proton linac field can be found in the 1968's linac conference round table discussions. The first presentations of "Equipartition" as a rule to avoid emittance growth and halo formation came more than ten years after (see [3] for more historical details and associated references). This has been done even though nobody demonstrates, even discusses, the applicability of the EQP theorem !

The "linac beam EQP rule" has been formulated assuming that the total energy spread of the particles in the bunch is equally shared between the transverses and longitudinal phase planes (x, x'), (y, y') and (z-z_s, z'), then assuming that equality (2) could be used.

The two transverse motions being usually exactly the same in a linac (same symmetry for the accelerator components, same emittances, same tunes…), the EQP relation has been expressed by the equality between the rms energy spread in one transverse phase-plane and the rms energy spread in the longitudinal phase-plane

$$E_{x\_rms} = E_{z\_rms} \quad (3)$$

This is a second assumption since going from the equality of the mean values (as stated by the EQP theorem) to the equality of the rms values is true only if the transverse and longitudinal distributions are the same, an obviously not realistic hypothesis.

To arrive at the linac EQP formula in terms of rms emittances (ε_x, ε_z), rms beam sizes (a in the x direction, b in the z direction) and rms phase advances per unit of time (ω_x, ω_z), the linear "rms particle" equation of motion must be used

$$d^2x_{rms}/dt^2 + \omega_x^2\, x_{rms} = 0$$

$$\rightarrow\ a' = \omega_x\, a \quad \text{(rms velocity in the x direction)}$$

$$\rightarrow\ E_{x\_rms} = 1/2\ m\, a'^2 = 1/2\ m\, \omega_x^2\, a^2$$

$$\rightarrow\ \varepsilon_x = \pi\, a\, a' = \omega_x\, a^2$$

with identical equations in the longitudinal phase plane. The EQP rule is easily derived from these equations

$$\frac{a}{b} = \frac{\varepsilon_x}{\varepsilon_z} = \frac{\sigma_z}{\sigma_x} \quad (4)$$

where the phase advances per unit of time ω have been replaced by the phase advances per lattice σ.

WHERE IS THE SWINDLE ?

Forgetting the confusion between mean and rms values, it is clear that the main equipartitionist' mistake comes from the fact that they apply the EQP theorem to systems which are obviously and hopefully not ergodic. Linac designers always choose parameter sets leading to safe working points with quite smooth and regular particle trajectories, even with severe tune depressions (sometimes up to 0.5), and even when the EQP rule is not respected !

Beam dynamics simulations show that most of the particles starting in the beam core will remain in the core and that most of the particles starting with large amplitudes do not end in the beam core. Realistic linac designs lead to the "weak perturbation regimes" described in the first section. Phase-space is mainly inhabited with slightly deformed non-resonant trajectories and the chaotic trajectories occupy limited and confined areas. The behaviour of the particles cannot be described by the equation (1), our beams stay out of the EQP theorem validity limit.

We can recall here that, as discussed in [3], the new techniques developed to characterize the level of disorder present in such nonlinear Hamiltonian systems could be very useful to provide reliable criteria to characterize the stability level of our beams as well as to optimize our accelerator designs.

Coupling Resonances

Mischievous spirits could say that, even if it is true that our particles are not subject to multiple collisions leading to energy exchanges at the microscopic level, hence energy equipartition at the macroscopic level, the coupling resonances which are well known to induce emittance exchanges in our accelerators are the source of EQP justifying the EQP rule (4).

We will demonstrate hereafter that it is a serious mistake to believe such an assertion, showing that energy exchanges induced by coupling resonances can occur although the EQP rule is respected and that tunes selected to avoid the coupling resonances can be found although the EQP rule is not respected. In other worlds we will see that, generally speaking, the EQP rule (4) has nothing to do with the coupling resonances.

This will demonstrate that linac designers must consider the coupling resonances as ring physicists do, and not as phenomena which lead to equipartition. Again, to think that the application of the EQP rule allows to avoid the coupling resonances is a serious mistake !

WHY THE FORMULA PRESENTLY USED TO DEFINE EQP IS WRONG ?

Just for fun, let's go further by accepting the fake idea that the EQP theorem can be restricted to 3 degrees of freedom leading directly to the equality (2) in the case of N particles confined in a box (cf. first section).

Remembering that the EQP theorem states that the energy sharing occurs only between the system' independent degrees of freedom, we must notice that equality (2) is valid only when the two transverse degrees of freedom are considered as independents, what they are manifestly not in most of our linacs (same emittances, same tunes).

Considering a total correlation between the two radial degrees of freedom, the EQP rule should be applied to the

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total radial mean energy and the longitudinal mean energy, then (2) rewritten

$$<\nu_r^2> = <\nu_x^2> + <\nu_y^2> = 2 <\nu_x^2> = <\nu_z^2>$$

leading to (3) rewritten $E_{x\_rms} = E_{z\_rms} / 2$ and (4) rewritten

$$\frac{a}{b} = \frac{\sqrt{2}\,\varepsilon_x}{\varepsilon_z} = \frac{\sigma_z}{\sqrt{2}\,\sigma_x} \quad (5)$$

WHY ENERGY EXCHANGES CAN OCCUR ALTHROUGH THE EQP RULE IS RESPECTED ?

The answer is straightforward : just because the EQP rule (4) which can be rewritten

$$\nu_z = (\varepsilon_x / \varepsilon_z)\, \nu_x \quad (4')$$

is not a means to avoid the coupling resonances usually defined by

$$n_1 \nu_x + n_2 \nu_z = p$$

with

n_1, n_2 and p integers,

n_1 and $n_2 \neq 0$ (coupling resonances), $p \geq 0$,

ν_x and ν_z the wave numbers ("tunes", $\nu = \sigma / 2\pi$) in the (x, x') and (z, z') phase planes respectively.

($N = |n_1| + |n_2|$ is the order of the resonance)

To illustrate this evidence, we can take the example of a beam characterized by

$$\varepsilon_x / \varepsilon_z = 0.85,\ \sigma_x / \sigma_{0x} = 0.75 \text{ and } \sigma_z / \sigma_{0z} = 0.70 \quad (6)$$

Following (4), the "equipartitionists" will add

$$\varepsilon_x / \varepsilon_z = \sigma_z / \sigma_x = 0.85 \quad (7)$$

Considering safe (< 90°) transverse phase advances per focusing period without space charge, this set of equations leads for example to a beam footprint characterized by

$$\sigma_{0x} = 80° \ (\nu_{0x} = \sigma_{0x} / 2\pi = 0.222)$$
$$\text{and } \sigma_x = 60° \ (\nu_x = \sigma_x / 2\pi = 0.167) \quad (8)$$
$$\sigma_{0z} = 73° \ (\nu_{0z} = \sigma_{0z} / 2\pi = 0.202)$$
$$\text{and } \sigma_z = 51° \ (\nu_z = \sigma_z / 2\pi = 0.142) \quad (9)$$

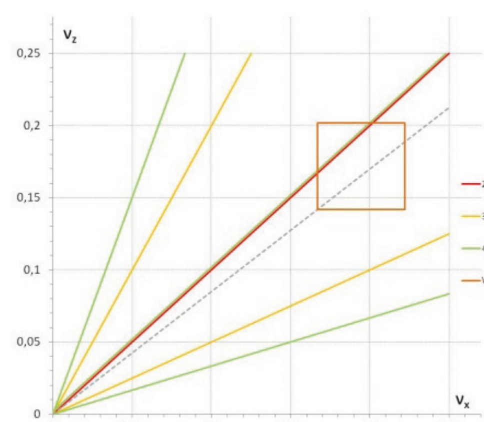

Figure 1 : Tune diagram with the coupling resonances up to the fourth order and beam footprint (brown) of the equipartitioned beam defined by (8) and (9).

As shown by Fig. 1 (brown footprint), such an "equipartitioned" beam will be subject to coupling resonances, then subject to emittances and energy exchanges !

More generally speaking, the EQP rule constraints the working point with space charge (σ_x, σ_z) or (ν_x, ν_z) to be on a line defined by (4') (the gray dashed line in Fig. 1) or (5). When the slope of this line (the emittance ratio $\varepsilon_x / \varepsilon_z$) is close to the slope of one of the coupling resonances, this resonance can affect the beam core. The closer the two slopes are, the more the coupling resonance will affect the beam core.

WHY SAFE TUNINGS CAN BE FOUND ALTHROUGH THE EQP RULE IS NOT RESPECTED ?

The answer is again straightforward : without the strong constraint of the EQP rule which dictates the choice of one tune when the other is fixed, the free choice of both radial and longitudinal tunes allows to select a working point which seats the beam footprint out of the coupling resonances.

To illustrate this point we can take the previous beam defined by (7) and keep the same longitudinal beam dynamics defined by (9), for example saying that it is the result of an optimization of the RF system and accelerating cavities. Then, the choice of the transverse focusing strength offered by the fact that we abandon the EQP constraint allows to push the beam away from the coupling resonances. The choice can be

$$\sigma_{0x} = 50° \ (\nu_{0x} = \sigma_{0x} / 2\pi = 0.139)$$
$$\text{and } \sigma_x = 36° \ (\nu_x = \sigma_x / 2\pi = 0.104) \quad (10)$$

In this case we have

$$\sigma_x\, \varepsilon_x / \varepsilon_z\, \sigma_z = 0.62$$

instead of 1 for an equipartitioned beam. The beam is then quite strongly non-equipartitioned but, as shown in Fig. 2 (blue footprint), is out of the coupling resonances which are the potential sources of energy exchanges.

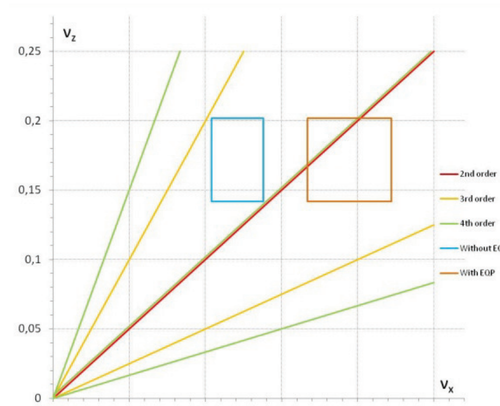

Figure 2 : Tune diagram with the coupling resonances up to the fourth order and the beam footprints : - brown for the equipartitioned beam of the previous section, - blue for the non-equipartitioned beam defined by (10) and (9).

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WHY SOME LINAC DESIGNERS NEVERTHELESS LIKE TO USE THE EQP RULE ?

A first answer can be “human beings like to believe in simple to understand and simple to put in practice ideas”. Obviously, the idea that the energy is shared between only 3 degrees of freedom is simple and the application of the EQP rule (4) is straightforward. It is more complex to consider the effect of the coupling resonances, to evaluate their level of excitation, to avoid them when they have significant bad effects…

A more subtle answer can be done making an analogy with the “Pascal’s gambit” [4] which can be summarized by “if I believe in God and there is no God then I have lost nothing, however if I don't believe in God and there is a God then I will go to hell, therefore it is rational to believe in God”. We have shown that such a transposition to the belief in EQP is a serious mistake since the application of the EQP rule imposes a strong constraint on the choice of the working point which can lead to a non-optimal beam dynamics and/or higher accelerator construction and operation costs, especially when the EQP rule implies the choice of RF systems and accelerating cavities which are not optimized.

Other interesting questions we will not discuss here could be

- Why the belief in EQP did not pollute the synchrotron world ?
- Why refereed papers promoting the use of the EQP rule have been / are still published ?

SUMMARY, TOPICS OF DISCUSSION

As said in the abstract, the purpose of this paper is to induce deep (and final ?) discussions on EQP in linac beams, ending with an answer to The question : “Equipartition, reality or swindle ?”.

In order to restrict these discussions on the most important points developed here, the judgment of the participants shall be focused on the following statements :

1- The linac beams are out of the EQP theorem validity limit, to apply the “EQP rule” designing a linac is a mistake.

2- The application of the “EQP rule” do not prevent emittance exchanges induced by coupling resonances.

3- Safe tunes with beam footprints out of the coupling resonances can be found when the “EQP rule” is not respected.

4- The constraint imposed by the “EQP rule” on a linac design can lead to a non optimized beam dynamics and higher construction and operation costs.

5- The question of energy exchange / emittance transfer must be analyzed as done in circular machines (tune diagram, evaluation of the resonance’ excitation strength).

6- The modern physics tools developed to characterize the level of disorder (chaos) present in nonlinear Hamiltonian systems could be applied to characterize and optimize our beams.

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