IMPEDANCE ESTIMATION BY PARABOLIC PARTIAL DIFFERENTIAL EQUATION FOR RECTANGULAR TAPER*

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Abstract

The mesh calculation based on the paraxial approximation can be much faster than ordinary methods when the bunch is very short. There are two reasons. One is to be able to choose the longitudinal mesh size independent of the bunch length. The other is that the problem can be solved as an initial-value problem in spite of frequency domain calculation.

However, the accuracy of the results by the approximation is not clear generally. It will be shown that the approximation is valid for rectangular tapered chamber in some frequency range.

INTRODUCTION

Recently, the calculation of wake field and the impedance has become more important because new accelerators require high current and much required fineness. In many cases they are usually calculated numerically by simulation using a mesh.

There are many methods of mesh calculation. The finite-difference time domain (FDTD) [1] and the finite integration technique (FIT) [2] are popular.

The mesh computation based on the paraxial approximation [3] can be much faster than ordinary methods if the bunch length is very short. The approximation has used in geometrical optics. Since several years ago, it has also used for beam field. The calculation of Coherent Synchrotron Radiation (CSR) by paraxial approximation give good results[4][5]. In Ref.[3], the analytical solution of geometric wake impedances by paraxial approximation are shown. They are that for axisymmetric geometry.

In these proceedings, numerical 3D calculation will be shown. The vertical impedance for rectangular tapered chamber is computed. It agrees with the analytic solution in the appropriate frequency.

The smaller angle taper should be better because the wave at small angle is dominant.

We will focus only on short range wake and completely conducting wall in this proceeding. Resistive wall is not considered. MKSA unit is used in these proceedings.

* Work supported by KEK and by Global COE Program "the Physical Sciences Frontier", MEXT, Japan.

IMPEDANCE

In these proceedings, special transformed fields for arbitrary function $\tilde{f}(x, y, z, t)$ are defined by

$$\tilde{f}(x, y, z, t) = \frac{1}{2\pi} \int \grave{f}(x, y, z, k) e^{-ik(ct-z)} dk. \quad (1)$$

Tilde means original value and 'grave' ($\grave{f}$) means the transformed value satisfying the equation above. This is like Fourier transform. However, the factor e^{ikz} has to be noticed.

The wake effect is able to be represented by impedance. Conventional vertical impedance is

$$Z_y = \frac{-i}{cqy_s} \int_{-\infty}^{\infty} dz \left[\grave{E}_y + c\grave{B}_x \right]_{\boldsymbol{r}_w = \boldsymbol{r}_s = (0, y_s)}, \quad (2)$$

where q is a charge, $\boldsymbol{r}_s = (x_s, y_s)$ is an offset of source particle, $\boldsymbol{r}_w = (x_w, y_w)$ is an offset of witness particle, y_s is small, and $\grave{E}_y, \grave{B}_x$ are the special transformed fields defined by Eq.(1). We will omit grave ($\grave{f} \to f$) from now on.

PARAXIAL APPROXIMATION

In this study, paraxial approximation is used. iIt is valid if the wave propagates at small angle θ from z axis, as shown in Figure 1.

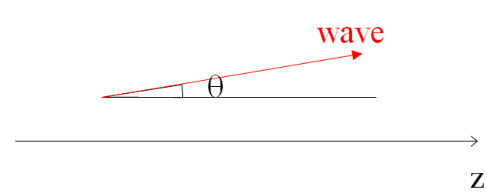

Figure 1: 'Paraxial' means the wave angle θ is small.

We will consider $c\tau \ll g$, where $\tau = (z_s - z_w)/c$ and g is the transverse minimum distance from beam axis. For rectangular chamber whose hight is smaller than width, g is the smallest half hight. In this range, backward or large angle propagating waves cannot take effect. Therefore, $\theta \ll 1$ can be assumed. Suppose $c\tau (\ll g)$ is bunch length, the waves of $\theta \geq 1$ which generated by a bunch don't catch up with the same bunch. When considering wake effect for bunch itself, paraxial approximation is valid for very short bunch length.

We divide electromagnetic fields into 2 terms as

$$\boldsymbol{E} = \boldsymbol{E}^{(b)} + \boldsymbol{\mathcal{E}}, \tag{3}$$

$$\boldsymbol{B} = \boldsymbol{B}^{(b)} + \boldsymbol{\mathcal{B}}, \tag{4}$$

where $\boldsymbol{E}$, $\boldsymbol{B}$ is net electric, magnetic field, and $\boldsymbol{E}^{(b)}$, $\boldsymbol{B}^{(b)}$ is vacuum fields, which is the solution of no wall. $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{B}}$ defined by above equation is the radiated fields.

The vacuum fields are found by solving Maxwell equation with charge density

$$\tilde{\rho}(\boldsymbol{r}, z, t) = q\delta(t - z/c)\delta(\boldsymbol{r} - \boldsymbol{r}_s). \tag{5}$$

Therefore

$$\tilde{\boldsymbol{E}}^{(b)}(\boldsymbol{r}, z, t) = \frac{q\delta(t - z/c)}{2\pi\epsilon_0 |\boldsymbol{r} - \boldsymbol{r}_s|^2}(\boldsymbol{r} - \boldsymbol{r}_s). \tag{6}$$

$$\tilde{\boldsymbol{B}}^{(b)} = \frac{1}{c}\boldsymbol{e}_z \times \tilde{\boldsymbol{E}}^{(b)} \tag{7}$$

From Eq(1),

$$\boldsymbol{E}^{(b)}(\boldsymbol{r}, z, k) = \frac{cq}{2\pi\epsilon_0 |\boldsymbol{r} - \boldsymbol{r}_s|^2}(\boldsymbol{r} - \boldsymbol{r}_s) \tag{8}$$

$$\boldsymbol{B}^{(b)} = \frac{1}{c}\boldsymbol{e}_z \times \boldsymbol{E}^{(b)} \tag{9}$$

Accordingly, Maxwell equation with respect to $\boldsymbol{\mathcal{E}}$ is

$$(\nabla_\perp^2 + 2ik\partial_z + \partial_z^2)\boldsymbol{\mathcal{E}} = 0. \tag{10}$$

The source term is cancelled.

We will show 3rd term of Eq.(10) is negligible. In time domain, $\tilde{\mathcal{E}}$ is superposition of the plain wave

$$\tilde{\boldsymbol{\mathcal{E}}}_{\eta,k}(x, y, z, t) = \hat{\boldsymbol{\mathcal{E}}}(\boldsymbol{\eta}, k)e^{-ik(ct - \eta_x x - \eta_y y - \eta_z z)}, \tag{11}$$

where $\|\boldsymbol{\eta}\| = 1$, $\hat{\boldsymbol{\mathcal{E}}}$ is a function of $(\boldsymbol{\eta}, k)$. η can be represented by

$$\boldsymbol{\eta} = \boldsymbol{e}_x \sin\theta\cos\phi + \boldsymbol{e}_y \sin\theta\sin\phi + \boldsymbol{e}_z \cos\theta. \tag{12}$$

θ is the angle of wave propagating from z axis. Neglecting the order of θ,

$$\boldsymbol{\eta} \approx \boldsymbol{e}_x\theta\cos\phi + \boldsymbol{e}_y\theta\sin\phi + \boldsymbol{e}_z(1 - \theta^2/2). \tag{13}$$

Therefore,

$$\begin{aligned} &\tilde{\boldsymbol{\mathcal{E}}}_{\eta,k}(x, y, z, t) \\ = \; &\hat{\boldsymbol{\mathcal{E}}}e^{-ik(-x\theta\cos\phi - y\theta\sin\phi + z\theta^2/2)}e^{-ik(ct - z)}. \end{aligned} \tag{14}$$

$\boldsymbol{\mathcal{E}}$ is the superposition of

$$\boldsymbol{\mathcal{E}}_\eta(x, y, z, k) = \hat{\boldsymbol{\mathcal{E}}}e^{ik(x\theta\cos\phi + y\theta\sin\phi - z\theta^2/2)}. \tag{15}$$

If θ^3 is not omitted but $O()$ is used,

$$\begin{aligned} &\boldsymbol{\mathcal{E}}_\eta(x, y, z, k) \\ = \; &\hat{\boldsymbol{\mathcal{E}}}e^{ik\left[(\theta + O(\theta^3))(\cos\phi x + \sin\phi y) - (\theta^2/2 + O(\theta^4))z\right]} \end{aligned} \tag{16}$$

Substituting it for $\boldsymbol{\mathcal{E}}$ in Eq.(10), it is found that the first and second term are order of $k^2\theta^2$ and the third term is order of $k^2\theta^4$. Accordingly, it is negligible. Eq.(10) is approximately the following equation,

$$\left[\nabla_\perp^2 + 2ik\partial_z\right]\boldsymbol{\mathcal{E}} = 0. \tag{17}$$

This is parabolic equation.

The parabolic equation and boundary conditions give transverse electric fields. From them, we obtain the other fields as

$$\boldsymbol{\mathcal{B}}_\perp \approx \frac{1}{c}\boldsymbol{e}_z \times \boldsymbol{\mathcal{E}}_\perp, \tag{18}$$

$$\mathcal{E}_z = \frac{ci}{k}\boldsymbol{e}_z \cdot (\nabla_\perp \times \boldsymbol{\mathcal{B}}_\perp), \tag{19}$$

$$\mathcal{B}_z = -\frac{i}{ck}\nabla_\perp \times \boldsymbol{\mathcal{E}}_\perp. \tag{20}$$

The impedances are calculated from these fields.

Paraxial approximation has two advantages. One is to be able to choose the transverse mesh size independent of the bunch length. The other is a problem can be solved as an initial-value problem in spite of frequency domain calculation. The reason is Eq.(10) is the first order with respect to z. Common frequency domain calculations are eigenvalue problem. Therefore, it must be faster than FDTD.

TAPERED RECTANGULAR CHAMBER

It is important to calculate the impedances of collimators because they are designed depending on the wake field. When the tapered angle of a collimator is small, bunch length is short, and the collimator is not axisymmetric, this impedance is hard to calculate by ordinary method. However, it can be fast calculated by paraxial approximation. Both small tapered angle and short bunch are good for paraxial approximation. When the tapered angle of a collimator is small, the wave at small angle is dominant. For short bunch length, only small τ is need if the wake in the same bunch is considered. Therefore, we computed the impedance for collimator which is not axisymmetric by paraxial approximation. As an example of the collimator, tapered rectangular structure was choiced.

Geometry

The half height $b(z)$ depends on z. The half width w is 10mm. Figure 2 is up side view of the calculated geometry from $-x$ direction.

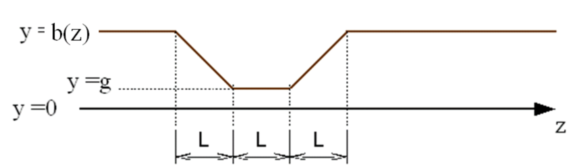

Figure 2: Side view of the upper half of the collimator. The maximum of $b(z)$ is 5 mm, and the minimum g is 3 mm. $L = 200$ mm.

Mesh

Figure 3 shows how to mesh about vertical and longitudinal direction. Mesh sizes $\Delta x,\ \Delta\eta,\ \Delta\zeta$ is constant. Here $\eta = y/b(z), \zeta = z/k$.

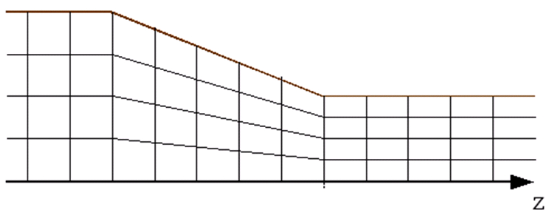

Figure 3: How to mesh.

Analytic Formula

Stupakov derived analytic formula,

$$Z_y(k \to 0) = -\frac{iwZ_0}{4}\int dz \frac{(b')^2}{b^3} \tag{21}$$

in Ref [6]. it is assumed $b \ll g \ll L$, and

$$k \ll \frac{b}{\alpha w^2}, \tag{22}$$

where α is the tapered angle.

Result

Figures 4 and 5 show comparisons of the simulation with the analytical value. In Fig. 4, the horizontal axis represents frequency f and the vertical axis represents real part of vertical impedance. Two kinds of data with different longitudinal mesh size are plotted. Blue diamonds show $\Delta\zeta = 10^{-9}\text{m}^2$. Green triangles show $\Delta\zeta = 10^{-8}\text{m}^2$. Orange line shows analytic solution. It is applicable while f is much smaller than 190 GHz because of condition (22). On the other hand, paraxial approximation requires that f is much greater than 16 GHz because $c\tau \ll g$ is corresponding to $f \gg c/(2\pi g) \approx 16$ GHz. Therefore, the simulation agrees with the analytic solution while 16 GHz $\ll f \ll$ 190 GHz although it fluctuates very much.

The imaginary impedance is shown in Fig. 5. Red squares show $\Delta\zeta = 10^{-9}\text{m}^2$. Purple $\times$'s show $\Delta\zeta = 10^{-8}\text{m}^2$. Yellow line shows analytic solution. As is mentioned, it is valid while f is much smaller than 190 GHz. On the other hand, paraxial approximation requires $f \gg 16$ GHz. As you can see, the simulation agrees with the analytic solution while f is applicable range.

CONCLUSIONS AND DISCUSSIONS

In summary, for rectangular tapered chamber, the vertical impedance by the simulation agrees with the analytic solution in the appropriate frequency. Accordingly, the simulation is effective when bunch length σ_z is much smaller than g.

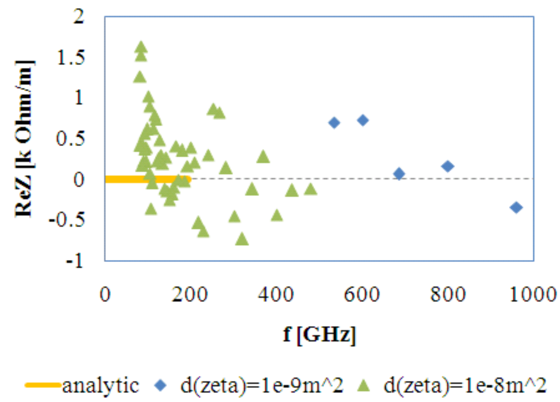

Figure 4: Real part of the impedance for rectangular taper.

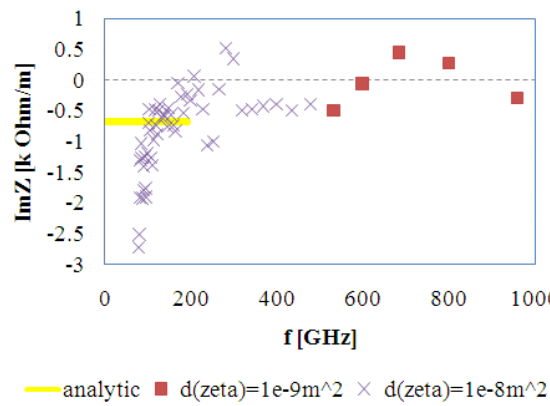

Figure 5: Imaginary part of the impedance for rectangular taper.

There are at least three future tasks. First is to improve the code. Second is to compare calculated impedances with that of other simulation codes, then to know how accurate and how fast the code is. Third is to calculate the impedance of ILC collimator. It is hard to calculate.

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