SPACE OF MOTION INTEGRALS IN PROBLEMS ON SELF-CONSISTENT CHARGED PARTICLE DISTRIBUTIONS*

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Abstract

A new approach for investigation of self-consistent distributions for charged particle beam in magnetic field is presented. According to this approach, the space of motion integrals is introduced. Specifying charged particles density in the space of motion integrals provides a self-consistent distribution for charged particle beam under some conditions, which are formulated. This approach allows simple graphical representation of various distributions by means of a special diagram.

INTRODUCTION

In the particle beam physics, dynamics of a charged particle beam is commonly described by the Vlasov Equation [1]. Its solutions are called self-consistent distributions, because the particles move in a field which is created by them. To find solutions of the Vlasov equation is very complicated problem due to nonlocal nonlinearity of the equation. Despite of complexity, there are founded various solutions of the Vlasov equation for a charged particle beam in magnetic field. The most known solution is the Kapchinsky-Vladimirsky distribution [2]. Another well-known solution is the Brillouin flow [3]. Both of them are degenerate distributions. The so called waterbag distribution is an example of nondegenerate distributions [4, 5].

In the present work we formulate a new approach of investigation of self-consistent distributions of charged particles. This approach can be applied for all problems on self-consistent distributions, but in this work we apply it only for a charged particle beam in longitudinal magnetic field.

We follow ideas formulated in previous works of the authors [6-15]. The space of integrals of motion integrals is introduced, and particle distribution density is specified as a density in this space. Under some conditions, which are specified further, the phase density can be expressed through the distribution density in the space of motion integrals. This approach equally works for cylindrical longitudinally uniform beam propagating through longitudinal uniform magnetic field and for nonuniform beam in magnetic and electric fields that can vary along beam axis.

Making use of this approach gives a possibility to construct new solutions of the Vlasov equation. They can be obtained taking a linear combination of known distributions, for example, rigid rotor distributions [16-18]. New solutions can be also found using some integral equation.

PROBLEM FORMULATION

Consider stationary axially symmetric longitudinally uniform beam propagating in a uniform longitudinal magnetic field **B**. Assume that longitudinal velocity components of all particles $v_z = \beta c$ are the same, and that transverse velocity components much less than longitudinal. In this case the phase space is four-dimensional, and the Vlasov equation can be written in the form

$$\mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{e}{m\gamma}\left(-\frac{1}{\gamma^2}\frac{\partial u}{\partial \mathbf{x}} + e\mathbf{v}\times\mathbf{B}\right)\frac{\partial f}{\partial \mathbf{v}} = 0. \quad (1)$$

Here $f = f(\mathbf{x}, \mathbf{v})$ is the distribution function, $\mathbf{x}$ and $\mathbf{v}$ are particle position and velocity, e, m, γ are particle charge, mass and reduced energy, u is self potential of the beam satisfying to the Poisson equation and the boundary conditions

$$\Delta u = -\frac{\varrho}{\varepsilon_0}, \quad u(0) = 0, \quad du/dr|_{r=0} = 0. \quad (2)$$

under assumption that beam is propagating in coaxial tube or in empty space. Further, ϱ is particle density in the configuration space, normalized as follows

$$\int f(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} = \int \varrho(\mathbf{x})\, d\mathbf{x} = \frac{I}{e\beta c}, \quad (3)$$

where I is the beam current.

Considering differential equations for particle trajectories in the self field of the beam, we obtain from one of them that

$$M = r^2(\dot\varphi + \omega_0), \quad (4)$$

is conserves along the trajectories. The other equation takes the form

$$\frac{d\dot r}{dt} = -\omega_0^2 r + \frac{M^2}{r^3} - \varepsilon\frac{\partial U}{\partial r}, \quad (5)$$

Equation (5) has the integral

$$H = \dot r^2 + \omega_0^2 r^2 + \frac{M^2}{r^2} + 2\varepsilon U. \quad (6)$$

Here $\omega_0 = eB_z/(2m\gamma)$ $\varepsilon = e/(m\gamma^3)$, M, H are azimuthal component of momentum and energy of transverse motion with an accuracy up to multiplier. Integral M is well known as the Bush integral.

The particle radial motion is shown in Fig. 1. Line 2 represents effective potential function

$$V_M(r) = M^2/r^2 + \omega_0^2 r^2 \quad (7)$$

for a particle with $M \neq 0$. If H is given, motion is possible only on the segment $[r_{\min}, r_{\max}]$, corresponding to this M, H.

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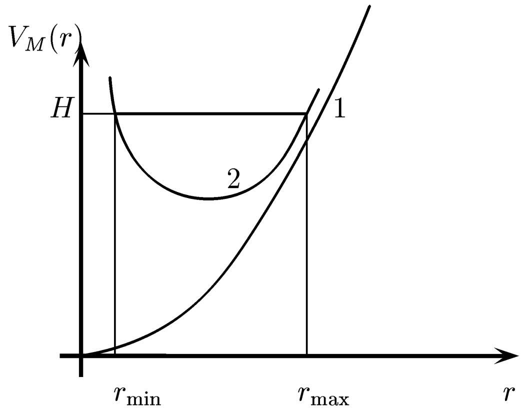

Figure 1: Radial motion of particles.

Line 1 represents effective potential function corresponding to $M = 0$. It can be shown that if the function $V_0(r)$ is strictly convex then the particle trajectories are confined and for each pair of admissible values of M and H there exists a unique radial trajectory. In configuration space, a radial trajectory corresponds to a set of particle trajectories such that any trajectory of this set can be obtained by rotating of any other trajectory by some angle.

THE SPACE OF INTEGRALS OF MOTIONS

Let us introduce the space of integrals of motion Ω_R for a cylindrical beam with radius R, as such set of values of the integrals of motion M, H that corresponding particle trajectory does not go out the boundary of the beam. That means that for all particles the inequality $r \le R$ holds.

It can be shown that the space of integrals of motion Ω_R is defined by the inequalities

$$\min_r V_M(r) < H \le M^2/R^2 + \omega_0^2 R^2 + 2\varepsilon U(R). \quad (8)$$

and $|M| < M^*$, where M^* is such M that the left and the right hand sides in inequality Eq. (8) are equal.

The set $\Omega(r)$ of admissible values of M and H such that trajectory with these M and H passes through points with coordinate r is defined as follows

$$\frac{M^2}{r^2} + \omega_0^2 r^2 + 2\varepsilon U(r) \le H \le \frac{M^2}{R^2} + \omega_0^2 R^2 + 2\varepsilon U(R). \quad (9)$$

$$|M| \le rR\sqrt{\omega_0^2 + 2\varepsilon \frac{U(R) - U(r)}{R^2 - r^2}}. \quad (10)$$

The space of integrals of motion Ω_R for radially confined beam and the set $\Omega(r)$ are represented on Figure 2. Upper and the low bounds of Ω_R Eq. (8) are marked as 1 and 2.

PARTICLE DISTRIBUTION DENSITY

In the previous section we described the particle distributions with making use of the distribution function. Commonly, it is defined as number of particles per a unit of the phase volume.

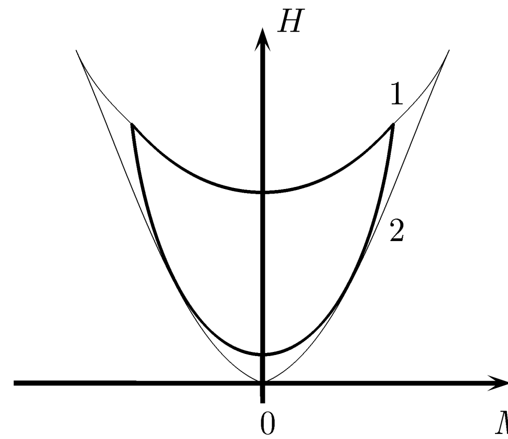

Figure 2: The space of integrals of motion Ω_R and the set $\Omega(r)$.

Let us introduce the concept of particle distribution density following the approach suggested in works [19, 20] As distinct from the distribution function, the definition of the distribution density does not require the notion of the phase volume.

Firstly, consider the nondegenerate case. Take a cell in a region occupied by particles. Let the edges of this cell can be described by increments of coordinates in this region δq^i $i = 1, \ldots, m$ where m is dimension of the region. Then the value that put in correspondence to this cell a number of particles in it will be called the particle density. From mathematical point of view, such value is differential form. It can be written in the form $n = n_{1,\ldots,m}\, dq^1 \ldots dq_m$ (product of differentials is regarded as external product).

For example, if we consider density of particles in the phase space (the phase density) then m is dimension of the phase space, and q^i are coordinates in it.

When particles are distributed on some surface in the phase space, their distribution is degenerate, because in this case the concept of the distribution function cannot be applied. Degenerate distributions can be described by forms of lower degree, which can be defined analogously to the form of top degree, corresponding to a nongenerate distribution.

The phase density satisfies to the Vlasov equation in the covariant form

$$n(t + \delta t, F_{w\,\delta t} q) = F_{w\,\delta t} n(t, q), \quad (11)$$

which was written down in the works [19, 20]. Here $F_{w\,\delta t}$ denotes the operation of Lie dragging along the vector field w defined by particles trajectories by the parameter t increment δt. If phase density is described by a form of top degree Equation (11) can be rewritten in the form

$$\frac{\partial n}{\partial t} = -\mathcal{L}_w n(t, q). \quad (12)$$

Here $\mathcal{L}_w$ denotes the Lie derivative along the vector field w.

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PARTICLE DISTRIBUTIONS IN THE SPACE OF INTEGRALS OF MOTION

Let us introduce the particle distribution density in the space of integrals of motion and $f(M,H)\,dM\,dH$ (further we use the letter f for designation of this density, instead of distribution function). Under conditions formulated in previous section, such distribution corresponds to some distribution in the phase space. To find the relation between them, let us note that M and H can be considered as coordinates in the phase space. Azimuthal angle φ and the phase of a particle on the trajectory θ can be regarded as other two coordinates. As it was assumed, particles are uniformly distributed on φ and θ. Therefore,

$$n_{\varphi\theta MH} = \frac{f(M,H)}{4\pi P(M,H)},$$

where $P(M,H)$ is:

$$P(M,H) = \int\limits_{r_{min}(M,H)}^{r_{max}(M,H)} \frac{dr}{|\dot{r}|} =$$

$$= \int\limits_{r_{min}(M,H)}^{r_{max}(M,H)} \frac{dr}{\sqrt{H - \omega_0^2 r^2 - M^2/r^2 - 2\varepsilon U(r)}}. \qquad (13)$$

Substituting $\tilde{n} = n_{\varphi\theta MH}$ to the Equation (12), we get

$$\frac{\partial\tilde{n}}{\partial t} = -\frac{\partial\tilde{n}}{\partial\varphi}\dot{\varphi} - \frac{\partial\tilde{n}}{\partial\theta}\dot{\theta} - \frac{\partial\tilde{n}}{\partial M}\dot{M} - \frac{\partial\tilde{n}}{\partial H}\dot{H} = 0.$$

It means that uniformity of the distributionon on the trajectory phases θ ensures its stationarity.

To compute the particle density in the configuration space, take into account that

$$n_{xyMH} = n_{\varphi\theta MH}\cdot det\left|\left(\frac{\partial(\varphi,\theta)}{\partial(x,y)}\right)\right| = \frac{n_{\varphi\theta MH}}{r|\dot{r}|}.$$

Then we get that

$$\varrho(r) = 2\int\limits_{\Omega(r)} n_{xyMH}\,dM\,dH =$$

$$= \frac{1}{2\pi r}\int\limits_{\Omega(r)} \frac{f(M,H)\,dM\,dH}{P(M,H)(H - M^2/r^2 - \omega_0^2 r^2 - 2\varepsilon U(r))^{1/2}}. \qquad (14)$$

It can be obtained also that the particle distribution function is

$$n_{xy\dot{x}\dot{y}} = \frac{f(M(\mathbf{x},\mathbf{v}),H(\mathbf{x},\mathbf{v}))}{2\pi P(M(\mathbf{x},\mathbf{v}),H(\mathbf{x},\mathbf{v}))}. \qquad (15)$$

In many problems specifying of component $n_{xy\dot{x}\dot{y}}$, which is distribution function, is more convenient because Integral (14) can be taken in analytical form.

For example, if consider the distribution with uniform phase density $n_{xy\dot{x}\dot{y}} = n_0$. specified in Ω_R under sufficient condition

$$H \le H_0 = \omega_0^2 R^2 + 2\varepsilon u(R), \qquad (16)$$

we obtained the "waterbag" distribution [4, 5]. In this case the Poisson equation takes the form

$$\frac{1}{r}\frac{dU}{dr}r\frac{dU}{dr} = -\frac{\pi e f_0}{\varepsilon_0}\Big(\omega(R^2 - r^2) + 2\varepsilon(U(R) - U(r)\Big).$$

Its solution can be expressed through the modified Bessel function I_0. Beam profile for this distribution is described by the expression

$$\varrho(r) = \varrho_B\left(1 - \frac{I_0(\sqrt{\lambda}r/R)}{I_0(\sqrt{\lambda})}\right), \quad \lambda = 2\pi e n_0 \varepsilon R^2/\varepsilon_0,$$

where $\varrho_B = \varepsilon_0 B_z \gamma/2m_0$ is density of the Brillouin flow which will be considered further.

Generalization of the waterbag distribution was considered in works [6, 10, 12–15]. This distribution has uniform density component $n_{xy\dot{x}\dot{y}}$ in all space Ω_R without restriction $H < H_0$.

Wide classes of self-consistent distributions can be obtained with making use of the density inversion theorem [21] which can be formulated as follows. Let $n_{xy\dot{x}\dot{y}}$ depends only on H, particles are situated inside the surface (16) in Ω_R, and $\varrho(r)$ is monotonic decreasing function.

Then the phase density can be found according to the expression

$$n_{xy\dot{x}\dot{y}}(V_0(r)) = -\frac{1}{dV_0/dr}\cdot\frac{d\varrho}{dr}, \qquad (17)$$

where $V_0(r)$ is the function defined by Eq. (7).

UNIFORMLY CHARGED BEAM

Let us find such phase distributions that particle density in the configuration space is uniform inside the beam cross-section $\varrho_{xy}(r) = \varrho_0$, $r \le R$. Then the Poisson equation yields $U(r) = -e\varrho_0 r^2/4\varepsilon_0$.

Firstly, consider the case when particle are distributed on the two-dimensional surface $M = 0$, $H = 0$, and assume that the phase density does not depend of φ. In this case the phase density is described by the form of the second order defined on the surface. As particle always lie on this surface it can be regarded as a phase space and the Vlasov Equation (11) can be written in the form (12):

$$\frac{\partial n_{r\varphi}}{\partial t} + \dot{r}\frac{\partial n_{r\varphi}}{\partial r} + \dot{\varphi}\frac{\partial n_{r\varphi}}{\partial\varphi} = 0.$$

This equation is satisfied as the distribution is stationary, $\dot{r} = 0$, and particles are evenly distributed on φ.

The solution under consideration corresponds to wide known Brillouin flow [3], when particle rotates around beam axis with the same angular velocity $\dot{\varphi} = -\omega_0$. As can be seen from (6), $\varrho_B = 2\varepsilon_0\omega_0^2/(e\varepsilon) = \varepsilon_0 B_z\gamma/(2m)$ is the spatial density of the Brillouin flow. In what follows, it is assumed that $\varrho_0 < \varrho_B$.

For uniformly charged beam the inequalities defining ω_R Eq. (8) take the form

$$2\omega|M| < H \le M^2/R^2 + \omega^2 R^2,$$

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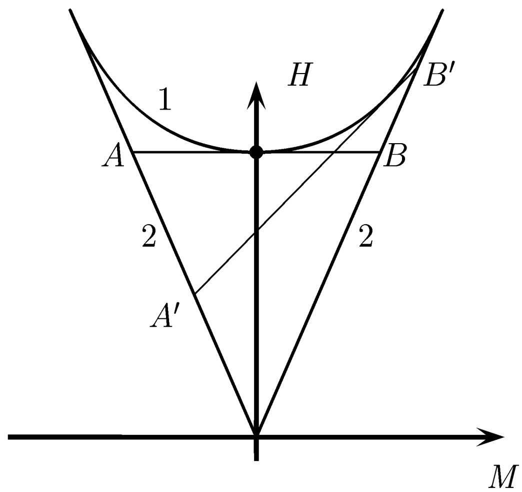

Figure 3: The space Ω_R for a uniformly charged beam.

where $\omega^2 = \omega_0^2 - e\varrho_0\varepsilon/(2\varepsilon_0)$ (see Fig. 3).

Let all particles are uniformly distributed on the straight line segment S_k, which is tangent to upper boundary of the set Ω_R :

$$S_k : H = kM + H_0, \quad H_0 = R^2(\omega^2 - k^2/4),$$

$|k| < 2\omega$, $(M, H) \in \Omega_R$ (segment $A'B'$ on Fig. 1). In this case, the particle density in the space of integrals of motion is described by the differential form of the first degree $f_0 dM$, $f_0 > 0$. In the phase space, density of such distribution is described by the form of degree 3 defined on a three dimensional surface corresponding to segment S_k. Analogously to previous case, we get

$$n_{\varphi\theta M} = \frac{f_0}{4\pi P(M, H)}, \qquad n_{xyM} = \frac{n_{\varphi\theta M}}{r|\dot{r}|}.$$

Then spatial density does not depend of r :

$$\varrho_0 = 2\int_{M_1}^{M_2} n_{xyM}\, dM = \frac{\omega f_0}{\pi} = const.$$

Here M_1, M_2 are roots of the denominator in the integrand.

This distribution is known as the rigid rotor distribution [16–18]. Here we offer simple geometrical representation of it in the space of integrals of motion.

At $k = 0$ (segment AB on Fig. 1), this distribution represents wide known Kapchinsky-Vladimirsky distribution [2], for which all particles are uniformly distributed on the segment AB (Fig. 1).

All distributions corresponding to various k give uniformly charged beam with the same radius R. Therefore any linear combinations of these distributions

$$f(M, H) = \sum \alpha_k f_k(M, H), \tag{18}$$

or their integral on the parameter k

$$f(M, H) = \int_{-2\omega}^{2\omega} f_k(M, H)\, dk, \quad f_k > 0. \tag{19}$$

give the uniform charged beam with radius R.

As an example of nontrivial distribution which can be obtained as integral, cite the distribution

$$f(M, H) = \frac{\pi\varrho_0}{2\omega^2(M^2 - HR^2 + \omega^2 R^4)^{1/2}}.$$

This nondegenerate distribution was found for the first time in the work [6]. It was mentioned also in the work [22].

Wide classes of self-consistent distributions can be found if Poisson equation is regarded as integral equation for $f(M, H)$ [10-15].

LONGWISE NONUNIFORM BEAM

Consider stationary azimuthally symmetric beam in longitudinal magnetic field in which all particles have the same longitudinal velocity v_z. Let R and ω_0 slow change along beam axis: $d\omega_0/dz \ll \omega_0/R$. Assume also that the spatial density is uniform within each cross-section: $\varrho_{xy} = \varrho_0(z)$, $r < R$.

In this case, M is also integral of motion. To get another integral, consider equation of radial motion

$$\frac{d^2 r}{dt^2} = -\omega_0^2 r + \frac{M^2}{r^3} + \lambda\frac{r}{R^2}. \tag{20}$$

Assume that at some instance particles lie inside an ellipse $r^2/a_0^2 + \dot{r}^2/c_0^2 = 1$, and that the beam envelope is defined only by particles with $M = 0$. Then it can be shown [8-14] that equation for the beam envelope $R(z)$ has the form [23]

$$\frac{d^2 R}{dt^2} = -\omega_0^2 R + \frac{\lambda}{R} + \frac{a_0^2 c_0^2}{R^3}, \tag{21}$$

which holds under assumption that at initial instance particles lie inside the ellipse $r^2/a_0^2 + \dot{r}^2/c_0^2 = 1$ in the phase space of the transverse motion. Here $\lambda = eJ/(2\pi\varepsilon_0 m\gamma^3 v^z)$.

It is easy to show that the system of Equations (20), (21) is particular case of the generalized Ermakov system, considered in the work [24]. Using the expression for its integral [24], one can show that the value

$$I = (\frac{dq}{d\tau})^2 + \frac{M^2}{q^2} + a_0^2 c_0^2 q^2. \tag{22}$$

is an integral of motion. Here $q = r/R$, $d\tau = ds/R^2$. The integral (22) was introduced for the first time in the work [8] and after that was used in the works [9-14] for description of self-consistent distributions for a charged particle beam. When $M = 0$, integral (22) coincides with the Courant-Snyder invariant [25], which is well-known in charged particle beam physics, and which is integral for the Ermakov system also [26]. Thus, instead of integral H, we introduced another integral of motion I, which depends not only from motion of a particle, but from motion of a beam as a whole.

As previously, let us introduce the space of integrals of motion and denote it by $\tilde{\Omega}_1$. It is easy to see that $\tilde{\Omega}_1$ is determined by inequalities

$$2a_0 c_0|M| < I \leq M^2 + a_0^2 c_0^2, \tag{23}$$

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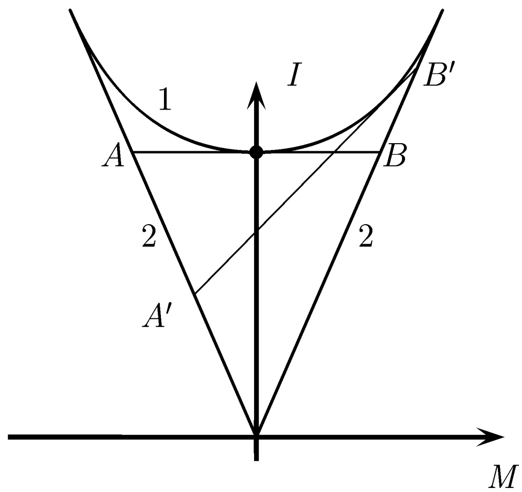

Figure 4: The space Ω_R for a uniformly charged beam.

and, therefore, looks like the set Ω_R for radially confined beam on Fig. 1, where H should be replaced by I.

Consider a particle distribution of some thin layer moving along beam axis. The phase space is four-dimensional, and M, I, φ and θ can be taken as coordinates. As previously, assume that particle uniformly distributed on phases θ and azimuthal angle φ.

At first, consider a case when particles are distributed on the two-dimensional surface $M = 0$, $I = 0$. Equation (12) yields

$$\frac{\partial n_{q\varphi}}{\partial t} + \dot{q}\frac{\partial n_{q\varphi}}{\partial q} + \dot{\varphi}\frac{\partial n_{q\varphi}}{\partial \varphi} = 0.$$

Therefore, such distribution is a stationary solution of the Vlasov equation. From physical point of view, it corresponds to a beam with radius changing along beam axis according to Equation (21), and rotating in each cross-section with angular velocity that also depends on z. Such distribution is analogue to the Brillouin flow, and can be called the generalized Brillouin flow.

Consider also a distribution when all particles are uniformly distributed on the segment S_k, which is tangent to upper boundary of the set $\tilde{\Omega}_1$:

$$S_k : I = kM + I_0, \quad I_0(k) = a_0^2 c_0^2 - k^2/4,$$

$|k| < 2a_0c_0$, $(M, I) \in \tilde{\Omega}_1$ (segment $A'B'$ on Fig. 1). Describe the particle density in the space of the integrals of motion by the differential form of the first degree $f_0 dM$, $f_0 > 0$. In the initial four-dimensional phase space such density is described by the form of degree 3 defined on the segment S_k. Analogously to the previous case, we get

$$n_{\varphi\theta M} = \frac{f_0}{4\pi P(M, I)}, \qquad n_{\tilde{x}\tilde{y}M} = \frac{n_{\varphi\theta M}}{q|\dot{q}|},$$

where $\tilde{x} = x/R$, $\tilde{y} = y/R$,

$$P(M, I) = \int_{q_{\min(M,I)}}^{q_{\max(M,I)}} (I - \frac{M^2}{q^2} - a_0^2 c_0^2 q^2)^{1/2}\, dq = \frac{\pi}{2a_0c_0}.$$

For spatial density we get

$$\varrho_{\tilde{x}\tilde{y}M} = \int_{M_1}^{M_2} n_{\tilde{x}\tilde{y}M}\, dM = \frac{a_0 c_0 f_0}{\pi} = const.$$

When $k = 0$ (segment AB on Fig. 1), we have analogue of the Kapchinsky-Vladimirsky distribution for nonuniform beam. It is easy to understand that taking a linear combination of such distributions with various k we also get a solution of the Vlasov equation.

Analogous approach can be also used for beam in external electric field [20, 27].

CONCLUSION

As a conclusion, list the main results of this work The spaces of integrals of motion are introduced for a longitudinally uniform beam and for nonuniform ("breathing") beam. The densities of particle distribution in these spaces are defined. The relations between these densities and the phase density, the spatial density, and the distribution function are established. It is shown that new self-consistent distributions can be obtained as linear combinations of the rigid rotor distributions. New integral I for longitudinally nonuniform beam is presented. New self-consistent distributions are also presented.

Self-consistent distributions written down in analytical form can be used as a beam models in optimization problems [14, 28, 29] and as test problems for beam simulation software.

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