REALISTIC APPROACH FOR BEAM DYNAMICS SIMULATION WITH SYNCHROTRON RADIATION IN HIGH ENERGY CIRCULAR LEPTON COLLIDERS

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Abstract

In extremely high energy circular lepton collider correct consideration of synchrotron radiation (SR) is important for beam dynamics simulation. We developed fast, precise and effective method to track the paricles in the lattice (including nonlinearities) when the radiation effects — classical damping and quantum emittance excitation — are distributed along the beam orbit. As an example we study beam dynamics in the FCC-ee lepton collider which is now under development at CERN [1]. Radiation effect on beam optics, dynamic aperture and momentum acceptance is discussed.

CONCENTRATED SR

Usual way to simulate SR in a circular lattice is to apply the following transformation to the coordinates of all particles once per turn at arbitrary azimuth s_0 [2] (the formulae are simplified for the case of flat lattice without betatron coupling)

$$\begin{aligned}
x &\mapsto a_x(x-\eta_x\delta)+\eta_x\delta+b_x\hat{r}_1\\
p_x &\mapsto a_x(p_x-\eta'_x\delta)+\eta'_x\delta+b_x(\hat{r}_2-\alpha_x\hat{r}_1)/\beta_x\\
y &\mapsto a_y y+b_y\hat{r}_3\\
p_y &\mapsto a_y p_y+b_y(\hat{r}_4-\alpha_y\hat{r}_3)/\beta_y\\
\delta=\Delta E/E_0 &\mapsto e^{-\frac{T_0}{2\tau_\delta}}\,\delta+\sigma_\delta\sqrt{1-e^{-\frac{T_0}{\tau_\delta}}}\,\hat{r}_5
\end{aligned}\quad, \qquad (1)$$

where

$$a_u=e^{-\frac{T_0}{2\tau_u}}\,,\qquad b_u=\sqrt{\varepsilon_u\beta_u\left(1-e^{-\frac{T_0}{\tau_u}}\right)},$$

E_0 — reference energy, T_0 — revolution period, τ_u — damping times ($u=x,y$), ε_u — emittances, β_u, α_u, η_x, η'_x — optical functions at s_0, and $\hat{r}_1\ldots\hat{r}_5$ — random values with standard distribution.

DISTRIBUTED SR

If the energy loss per turn is very large, then the technique described above may provide erroneous results. So, we developed an algorithm, which takes into account realistic distribution of SR along the lattice.

Radiated Energy

In a dipole magnet of the length L and bending angle θ an electron with relativistic factor γ follows an arc with radius $\rho=L/\theta$ and radiates amount of energy equal to

$$W_0=\frac{2\theta e^2}{3\rho}\gamma^4\,.$$

Spectral power density is the following

$$\frac{dW}{d\omega}=\frac{W_0}{\omega_c}S\left(\frac{\omega}{\omega_c}\right),\quad \text{where}\quad \omega_c=\frac{3c}{2\rho}\gamma^3,$$

or

$$\frac{dW}{dy}=\frac{W_0}{y}S(y),\quad \text{where}\quad y=\frac{\omega}{\omega_c}\,.$$

$S(y)$ is so called spectral function

$$S(y)=\frac{9\sqrt{3}}{8\pi}y\int_y^\infty K_{5/3}(t)dt\,.$$

Then spectral photon density can be written as follows

$$s(y)=\frac{dN}{dy}=\frac{1}{y}S(y)\,.$$

Mean number of photons emitted during single passage through the magnet is

$$\bar{N}=\frac{W_0}{\hbar\omega_c}\int_0^\infty s(y)dy=\frac{5\sqrt{3}}{6}\alpha\gamma\theta\,,$$

where α is the fine structure constant. Then average photon energy is

$$\bar{E}=\frac{W_0}{\bar{N}}=\frac{4\sqrt{3}}{15}\frac{\lambdabar_e}{\rho}E_e\gamma^3\,,$$

where E_e is the electron rest energy, $\lambdabar_e$ is the reduced electron wavelength.

All radiation acts are independent, hence the number of actually emitted photons N has Poisson distribution with the parameter $\bar{N}$. To obtain energy of the i-th photon one should generate random value y_i with the following distribution density function

$$f(y_i)=\frac{3}{5\pi}\int_{y_i}^\infty K_{5/3}(t)dt\,, \qquad (2)$$

where K is a modified Bessel function of the second kind. Such a distribution will be referred to as SR-distribution, notation $y_i\in SR$ means that y_i obeys this distribution (a method for generation of this distribution will be described in the next subsection). After emission of the i-th photon coordinate δ is changed by

$$\Delta_i\delta=-\frac{3\lambdabar_e}{2\rho}\frac{\gamma^3}{\gamma_0}y_i\,,\quad i=1\ldots N\,,$$

where γ_0 is the relativistic factor of the reference particle. It should be noted that

$$\frac{\gamma}{\rho}=\frac{e}{E_e}B\,,\quad \gamma=\gamma_0\,(1+\delta)\,,$$

ISBN 978-3-95450-136-6

where B is on-axis magnetic field. Then

$$\bar{N} = \frac{5\sqrt{3}}{6}\frac{\alpha e}{E_e}BL\,, \quad \Delta_i\delta = -\frac{3}{2}\frac{e\lambdabar_e}{E_e}\gamma_0 B(1+\delta)^2 y_i\,. \tag{3}$$

Particle's path in the bending magnet depends on its initial horizontal coordinate x_0 (at the entrance pole face) and pole face rotation angles. If the magnet has quadrupole field of strength $k_1 = \frac{1}{B\rho}\frac{\partial B}{\partial x}$, then off-axis particles travel in different magnetic field. To take these effects into account we make the following substitutions in (3)

$$\begin{aligned} &B \mapsto B\,(1 + k_1\rho x_0)\,,\\ &L \mapsto L\,(1 + x_0/\rho) - x_0(\tan\varphi_1 + \tan\varphi_2)\,, \end{aligned}$$

where φ_1, φ_2 are the rotation angles for the entrance and exit pole face of the dipole. We should also substitute δ in (3) by its value δ_0 at the entrance pole. Finally

$$\begin{aligned} \bar{N} &= \frac{5\sqrt{3}}{6}\alpha\theta\gamma_0\,(1 + k_1\rho x_0)\,(1 + h^* x_0)\,,\\ \Delta_i\delta &= -\frac{3\lambdabar_e}{2\rho}{\gamma_0}^2\,(1+\delta_0)^2\,(1 + k_1\rho x_0)\,y_i\,, \end{aligned}$$

where

$$h^* = \frac{1}{\rho} - \frac{\tan\varphi_1 + \tan\varphi_2}{L}\,.$$

Generation of SR-Distribution

Given the distribution density (2) and the integral representation of K-function

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh(\nu t)dt\,, \quad \mathrm{Re}\,(z) > 0\,,$$

we can find distribution function of SR-distribution

$$F(z) = 1 - \frac{3}{5\pi}\int_0^\infty \frac{\cosh\left(\frac{5}{3}t\right)}{\cosh^2 t}e^{-z\cosh t}dt\,.$$

$y \in SR$ can be generated using inversion method [3], its main idea is the following: if ξ has uniform distribution over $[0;1]$ segment, then $F^{-1}(\xi) \in SR$. We will use analytical approximation of $F^{-1}(\xi)$, which will be denoted as $\widetilde{F}^{-1}(\xi)$. Its asymptotics should be the same as for $F^{-1}(\xi)$. Given asymptotics for $F(z)$

$$F(z) \xrightarrow[z\to 0]{} const\cdot z^{1/3}\,, \quad F(z) \xrightarrow[z\to\infty]{} const\cdot\frac{e^{-z}}{\sqrt{z}}\,,$$

we may take the following expression for $\widetilde{F}^{-1}(\xi)$

$$\widetilde{F}^{-1}(\xi) = C\,(-\ln(1-\xi^a))^{3/a}\,.$$

So, instead of y we will generate $\widetilde{y}$, which has the distribution function $\widetilde{F}(z)$. Thus $\widetilde{F}(z)$ should be close to $F(z)$, this can be achieved by appropriate choice of C and a values. Let the first two moments of y be the same as for $\widetilde{y}$. Also we have an expression for the n-th distribution moment

$$\langle y^n\rangle = \int_0^1 \left(F^{-1}(\xi)\right)^n d\xi$$

and similar expression for $\langle\widetilde{y}^n\rangle$. Using computer algebra system Maple 9.5 [4], we obtain $C = 0.5770253543282$, $a = 2.535608814842$.

Table 1: Relative Deviation of the First Four Moments of Energy Distribution from Theoretical Value $\langle y^n\rangle$ for the simulation Techniques Proposed in the Present Paper ($\langle\widetilde{y}^n\rangle$) and the One Proposed in [5] ($\langle {y_t}^n\rangle$).

n	$\langle y^n\rangle$	$\Delta\langle\widetilde{y}^n\rangle$, %	$\Delta\langle {y_t}^n\rangle$, %
1	$8\sqrt{3}/45$	$-3\cdot 10^{-8}$	$-6\cdot 10^{-5}$
2	$11/27$	$1\cdot 10^{-7}$	-0.5
3	$224\sqrt{3}/405$	1.56	-1.8
4	$1309/405$	4.83	-5.1

Two another techniques for SR-distribution generation were proposed in [5]. The first of them also involves inversion method, but its accuracy is poor because $F(z)$ is approximated with an inversible function instead of direct approximation of $F^{-1}(\xi)$. The second one involves lookup table and has much better accuracy. Let $\langle {y_t}^n\rangle$ be the values of the first four distribution moments for the lookup table method from [5]. Table 1 summarizes relative deviations for $\langle\widetilde{y}^n\rangle$ and $\langle {y_t}^n\rangle$ from theoretical values $\langle y^n\rangle$. So, our method is significantly more accurate. From now on we will assume that $\widetilde{F}(z) \equiv F(z)$ and $\widetilde{y} \equiv y$.

Transversal Motion

Energy deviation due to SR photons emission affects particle's motion in the bending plane. In a flat lattice all bends are horizontal, hence x and p_x are to be changed along with δ. Radiation damping in the magnet in both transversal planes is proportional to the magnet's contribution to I_2 integral, squared quantum exitation amplitude is proportional to the contribution to I_{5x}. Equilibrium distribution of the horizontal coordinates is gaussian, so we can apply transformations (1) to x and p_x in each bending magnet separately, assuming that the addition due to quantum exitation in each magnet is also gaussian. So, all radiation acts in the magnet can be simulated at once at its exit pole face. Finally, the following transformation should be applied to the coordinates of each particle after tracking through each bending magnet

$$\begin{aligned} &x \mapsto e^{c_{1x}\Delta\delta}(x - \eta_x\delta) + \eta_x(\delta + \Delta\delta) + c_{2x}\hat{r}_1\sqrt{\Delta^2\delta}\,,\\ &p_x \mapsto e^{c_{1x}\Delta\delta}(p_x - \eta'_x\delta) + \eta'_x(\delta + \Delta\delta) + c_{2x}\tfrac{\hat{r}_2 - \alpha_x\hat{r}_1}{\beta_x}\sqrt{\Delta^2\delta}\,,\\ &y \mapsto e^{c_{1y}\Delta\delta}y\,, \quad p_y \mapsto e^{a_y\Delta\delta}p_y\,, \quad \delta \mapsto \delta + \Delta\delta\,, \end{aligned} \tag{4}$$

where

$$\begin{aligned} &\Delta\delta = \textstyle\sum_{i=1}^N \Delta_i\delta\,, \quad \Delta^2\delta = \sum_{i=1}^N (\Delta_i\delta)^2\,,\\ &c_{1x,1y} = \frac{3T_0}{2\tau_{x,y}r_e{\gamma_0}^3 I_2}\,,\\ &c_{2x} = \sqrt{\frac{24\sqrt{3}}{55}\frac{\varepsilon_x\beta_x\langle H_x\rangle}{\alpha{\gamma_0}^5{\lambdabar_e}^2 I_{5x}}\left(1 - e^{-\frac{T_0}{\tau_x}}\right)}\,, \end{aligned}$$

I_2, I_{5x} — radiation integrals, $\langle H_x\rangle$ — horizontal dispersion invariant averaged over the magnet, $\beta_x, \alpha_x, \eta_x, \eta'_x$ — horizontal optical functions at the exit pole of the magnet, $\hat{r}_1$,

$\hat{r}_2$ — random values with standard distribution. Quantum excitation in the vertical plane can be simulated once per turn, as in (1).

SAWTOOTH EFFECT AND TAPERING

Distributed energy losses lead to variation of equilibrium beam energy $\langle\delta\rangle$ along the lattice: it drops in bending magnets and rises in RF cavities. This is so called sawtooth effect, which also leads to the closed orbit distortions, $\Delta x_{co} \approx \eta_x \langle\delta\rangle$, therefore, reference particle becames off-axis in quadrupoles and higher multipoles. So, in high energy rings all the optics will be completely distorted, dynamic aperture and energy acceptance will drop significantly due to sawtooth effect. It can be cured by a variation of magnetic field in beamline elements in proportion to varying equilibrium energy (magnet tapering). To make optics in tapered and original lattice as close as possible, one should change steering field and multipole gradients in each beamline element in proportion to $(1 + \langle\delta\rangle)$. Then, to take into account the effect of the dipoles on the closed orbit, the following transformation should be applied to the horizontal coordinates after each dipole

$$
\begin{aligned}
x &\mapsto x + \rho(1 - \cos\theta)\,\Delta\langle\delta\rangle\,, \\
p_x &\mapsto p_x + \sin\theta\,\Delta\langle\delta\rangle\,,
\end{aligned}
$$

where $\Delta\langle\delta\rangle = W_0/E_0$ is the variation of equilibrium energy in the dipole.

SR FROM QUADRUPOLES

Particle follows curved trajectory and therefore emits SR photons not only in dipoles but also in other beamline elements. Additional energy loss due to this effect, averaged over beam particles, is small compared to total losses even for high energy rings. Hence radiation integrals and beam sizes stay unchanged, but coordinate dependent losses, especially in strong final focus quadrupoles, distort optics for particles with large amplitudes, which leads to decrease in dynamic aperture [6].

The simpliest way to study this effect is to consider each strong quadrupole as a "variable strength dipole" with parallel pole faces and no quadrupole gradient. This fictitious dipole acts in both transversal planes and has different bending angle and radius of curvature each turn for each particle. These values will be different for horizontal and vertical planes

$$
\theta_x = |p_{x1} - p_{x0}|\,, \quad \theta_y = \left|p_{y1} - p_{y0}\right|\,, \quad \rho_{x,y} = L/\theta_{x,y}\,,
$$

where p_{x0}, p_{y0} are the transversal momenta at the entrance pole face, p_{x1}, p_{y1} are the transversal momenta at the exit pole face of the quadrupole. So, radiation in both transversal planes should be simulated independently

$$
\begin{aligned}
\bar{N}_{x,y} &= \frac{5\sqrt{3}}{6}\alpha\theta_{x,y}\gamma_0\,, \quad N_{x,y} \in Poisson\left(\bar{N}_{x,y}\right)\,, \\
(\Delta_i\delta)_{x,y} &= -\frac{3\lambdabar_e}{2\rho_{x,y}}\gamma_0{}^2\,(1+\delta_0)^2\,y_i\,, \quad i = 1\ldots N_{x,y}\,,
\end{aligned}
$$

$$
\begin{aligned}
\Delta\delta &= \textstyle\sum_{i=1}^{N_x}(\Delta_i\delta)_x + \sum_{i=1}^{N_y}(\Delta_i\delta)_y\,, \\
\Delta^2\delta &= \textstyle\sum_{i=1}^{N_x}((\Delta_i\delta)_x)^2 + \sum_{i=1}^{N_y}\left((\Delta_i\delta)_y\right)^2\,.
\end{aligned}
$$

Then the transformation (4) should be applied.

SIMULATION RESULTS FOR FCC-ee

The simulation technique described above was implemented as part of TrackKing simulation program [7]. N-turn DA at the given azimuth is defined as 3D region in (x, y, δ) coordinates visited by particles, which survived N turns of tracking. Initial particle distribution is uniform over these 3 coordinates with other 3 zeroed and wide enough to span the whole stability region. DA can be plotted as 3 projections of this region.

FCC-ee is 100 km e+e- collider with beam energy 45–175 GeV. Simulations were performed for preliminary version of 175 GeV FCC-ee lattice with 4 different algorithms: without SR; with concentrated SR; with distributed SR and tapering; with distributed SR, tapering and SR from final focus quadrupoles. DA borders (in units of beam sizes) are shown in Fig. 1.

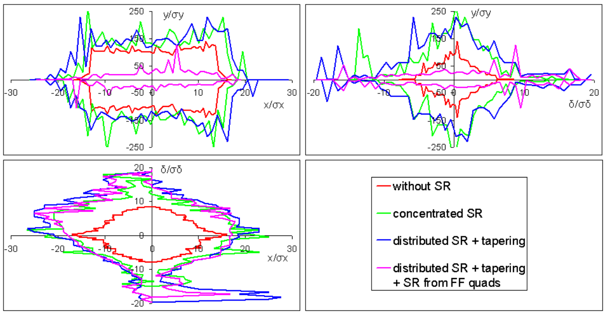

Figure 1: 500-turns DA of 175 GeV FCC-ee lattice.

Introducing SR into simulations increases energy acceptance considerably and also increases transversal DA slightly. At this point concentrated and distributed algorithms of SR simulation give similar results, but vertical DA decreases dramatically, when SR from quadrupoles is added (only distributed algorithm has this option). Further studies are required to explain the results correctly.

CONCLUSION

Dynamic aperture and energy acceptance of FCC-ee depends strongly on the choice of SR simulation technique, so, the most realistic one should be taken. The method described in this paper includes simulation of radiation damping and quantum excitation in longitudinal and both transversal planes. It contains procedure for precise generation of SR photons spectrum and takes into account realistic distribution of emission points along the lattice. The only assumption is that the addition to horizontal coordinates due to quantum excitation has gaussian distribution in each bending magnet. The important advantage of this method is the possibility of simulating SR from quadrupoles and studying of the magnet tapering options.

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