MAXIMUM ENTROPY TOMOGRAPHY RECONSTRUCTION

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Abstract

The goal of tomography is to reconstruct a higher dimensional distribution from a series of projections measured in a lower dimensional subspace. In the absence of large number of projections, the maximum entropy algorithm can reconstruct a distribution that maximizes the entropy and simultaneously reproduce all the measured projections exactly. The MENT algorithm [1] has been applied to the reconstruction of the transverse and longitudinal phase space distributions at particle accelerators. Only one- dimensional intensity profiles of different beam transfer matrices have to be measured. The article mainly completed the code for the tomographic reconstruction of the longitudinal phase space where non-linear transformations have to be taken into account, and finally introduced the test result of the real data taken from accelerator in TRIUMF for transverse tomography [3].

MAXIMUM ENTROPY TOMOGRAPHY

Tomography

We will only be dealing with 2-D distribution here. Let f(x,y) be the source distribution defined over an area. It satisfies

$$f(x,y) \geq 0 \quad and \quad \iint f(x,y)dxdy = 1 \qquad (1)$$

The projection P(x) of this distribution on the x-axis is defined by

$$\mathrm{P}(x) = \int_{-\infty}^{+\infty} f(x,y)dy \qquad (2)$$

The input data for tomographic reconstruction is a set of such projection onto N different s-axes defined by a set of transformation matrices

$$\begin{pmatrix} s \\ t \end{pmatrix} = R_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_i \begin{pmatrix} x \\ y \end{pmatrix} \qquad (3)$$

The transformation matrix Ri can be a rotation matrix, used for real space reconstruction, or the beam transport matrices for reconstruction of phase space. It conserves the area of the source distribution because $\det(R_i)=1$. Using the inverse transformation from the ith projection coordinates (s,t) back to the source plane (x,y), The ith projection is represented as

$$P_i(s) = \int_{-\infty}^{+\infty} f[x_i(s,t), y_i(s,t)]dt \qquad (4)$$

The goal is to invert Eq. (4) and determine the function f(x,y). However, the inversion is not unique unless the number of projections I is infinite. For a finite number of the measurements, many different distributions exist that can reproduce all the measured projections. Out of these distributions, the one that has the maximum entropy and satisfies the boundary conditions of Eq. (4) is the most appropriate one, because it contains the least information.

Maximum Entropy Algorithm

In the thermodynamics entropy is defined as a measure of the multiplicity of system. For the continuous distribution f(x,y), the entropy is written as

$$\mathrm{E}(f) = -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\ln f(x,y)dxdy \quad (5)$$

For a system of a large number of particles, the most probable distribution will be the distribution of the highest entropy.

Taking into account the boundary conditions, the extended entropy function is written as

$$(f,\lambda) = E(f) - \sum_{i=1}^{I}\int_{-\infty}^{+\infty} \lambda_i(s)[P_i - f(x_i,y_i)dt]ds \qquad (6)$$

Where x_i and y_i are functions of s,t, and where the $\lambda_i(s)$ denotes the Lagrange multiplier functions. The conditions for the stationary solution are

$$\frac{\partial\varepsilon(f,\lambda)}{\partial\lambda_i} = 0 \quad and \quad \frac{\partial\varepsilon(f,\lambda)}{\partial f} = 0 \qquad (7)$$

The first condition in Eq. (7) is in fact equivalent to the constraints defined by Eq. (4), whereas the second one gives

$$-\ln[f(x,y)] - 1 + \sum_{i=1}^{I}\lambda_i = 0 \quad or \quad f(x,y) = \prod_{i=1}^{I} H_i \qquad (8)$$

Where the unknown Lagrange multipliers λ_i have been replaced by the equally unknown function $H_i = \exp(\lambda_i - 1/N)$. The arguments of these functions are $s_i = a_i x + b_i y$, completely determined by the projection. So, the task is merely to find these H-values for the equation.

Since the measured projections are received as discrete rather than continuous distributions, it's natural to formulate a binned projection as follows

$$G_{ij} = \int_{s_{ij}}^{s_{i(j+1)}} P_i(s)ds = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\Gamma_{ij}dxdy \qquad (9)$$

Where the coordinate transformation from (s,t) to (x,y) with Jacobian equal to 1 is applied, and $\Gamma_{ij}(j = 1,2,\ldots,J_i)$ denotes a characteristic function

$$\Gamma_{ij}(s) = \begin{cases} 1 & s_{ij} \leq s \leq s_{i(j+1)} \\ 0 & otherwise \end{cases} \qquad (10)$$

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Therefore, Eq. (8) can be written as

$$f(x,y)=\prod_{i=1}^{I}\sum_{j=1}^{J_i}H_{ij}\Gamma_{ij} \tag{11}$$

Substituting Eq. (11) into Eq. (9) gives an iteration relation for the factors H_{ij}

$$G_{ij}=H_{ij}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dxdy\Gamma_{ij}\left\{\prod_{k\neq i}^{I}\sum_{l=1}^{J_k}H_{kl}\Gamma_{kl}\right\} \tag{12}$$

So

$$\mathrm{Hij}=\frac{G_{ij}}{\iint dxdy\Gamma_{ij}\{\prod_{k\neq i}^{I}\sum_{l=1}^{J_k}H_{kl}\Gamma_{kl}\}} \tag{13}$$

After the H-factors are computed, they can be substituted back into Eq. (11) to compute the distribution function f(x,y).

Longitudinal Phase Space Tomography

For longitudinal tomography, the final result we get is about ΔE and Δt

$$\begin{aligned}\Delta E &= \Delta E_i + qU_a(\cos(\phi)-\cos(\varphi_0))\\ &= \Delta E_i - qU_a\sin(\phi 0)(\phi-\phi 0)\\ &\quad -\frac{1}{2}qU_a\cos(\phi 0)(\phi-\phi 0)\,\hat{}\,2+\cdots\end{aligned} \tag{14}$$

where

$$\phi\text{-}\phi 0=2\pi f\Delta t$$

as we all know, Δt is not that easy to measure, but we can easily measure ΔE ,if we use the tomography reconstruction, we can easily get the distribution of the ΔE and Δt .

TEST RESULTS FOR LONGITUDINAL TOMOGRAPHY

Test Resulto of Longitudinal Tomography for Short Bunches

For short bunches, the ΔE we test is between -0.1 keV~0.4 keV, the Δt is between -0.4 ps~0.4 ps. We test the tomography program using 12 projections. Figure 1 is the original figure. And we use 12 different transformed matrixes to transform the figure.

And after we get the transformed figures, we do projections on x axis, and we get 12 different projections and 12 transformed matrixes corresponding to the projections, and these are the data we input to the tomography program. Figure 2 is the reconstructed figure we get from tomography. This result matches pretty well with the original input figure.

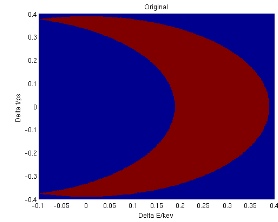

Figure 1: Original figure of short bunches.

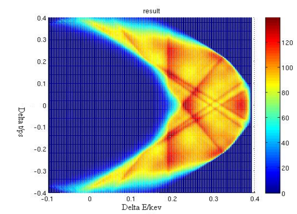

Figure 2: The reconstructed figure of short bunches.

Test Result of Longitudinal Tomography for Long Bunches

For long bunches, the ΔE we test is still between -0.1kev~0.4kev, but the Δt is between -40ps~40ps. As the same with the short bunches test, we still run the tomography using 12 projections. Figure 3 is the original figure. And we still use 12 different transformed matrixes to transform the figure. Figure 4 is the reconstructed figure we get from tomography. As we can see, this result match pretty well with the original input figure.

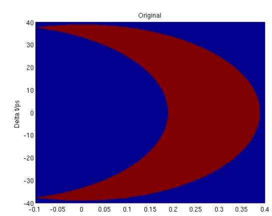

Figure 3: The original figure of long bunches.

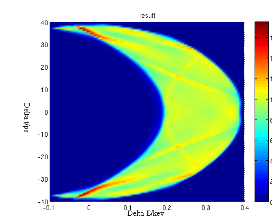

Figure 4: The reconstructed figure of long bunches.

Longitudinal Tomography for no Taylor Expansion

For real data, if the bunch of the particle is too long, then we cannot do Taylor expansion, because we do Taylor expansion of cos(phi) at phi0, and keep up to the 2nd order term of (phi-phi0) while omitting the other higher order terms. This implies that (phi-phi0) must be much smaller than 1 in magnitude, that means

$$\phi\text{-}\phi 0=2\pi f\Delta t<1$$

$$\Delta t=<\frac{1}{2\pi f}=\frac{1}{2\pi\times 0.0013\mathrm{THz}}=122\ \mathrm{ps}$$

Therefore, when the bunch is too long, the Taylor expansion cannot be properly used. So I improve the code, and change the definition of the matrix, and change the code of coordinate transformation. And finally we succeed solving the problem. And there is no Taylor expansion for long bunch. And I test the code, and it can work perfectly and the result of reconstruction is nearly the same as above we test.

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APPLICATION OF TRANSVERSE TOMOGRAPHY

The real data we taken from real machine is the viewscreen like Fig. 5, and we can change the work current of the magnet to change the shape of the beam bunches just like Fig. 6. We can use the viewscreen to record the different x-y shapes of the beam bunches.

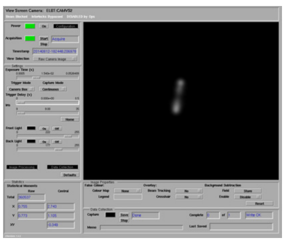

Figure 5: Viewscreen figure of bunches.

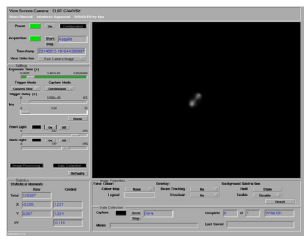

Figure 6: Viewscreen figure of bunches.

After processing the viewscreen figure, we can get the x-y figure of the beam. Then we can do projections on x axis and y axis. And the projections we input to the tomography are showed in Fig. 7 and Fig. 8

Figure 7 is projections on y axis, so we can use these data to get the result figure of y and y'. Figure 8 is projections on x axis, so we can use these data to get the result figure of x and x'.

Figures 9 and 10 are the results of y and y'. Figures 11 and 12 are the results of x and x'.

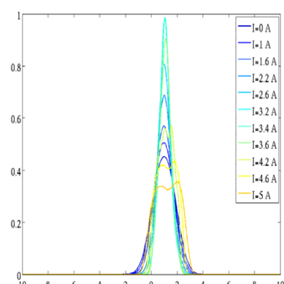

Figure 7: Projections on y-axis.

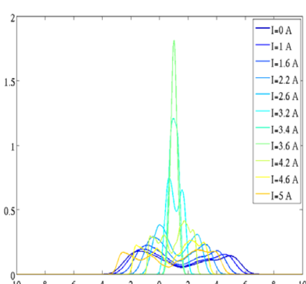

Figure 8: Projections on x-axis.

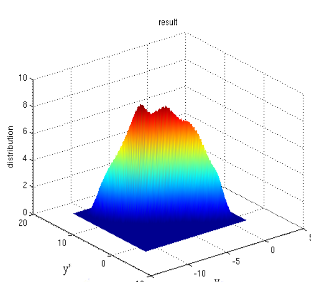

Figure 9: Results of y and y'.

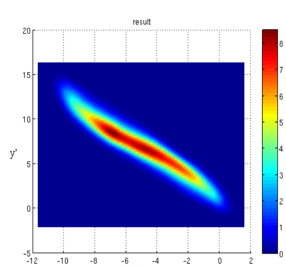

Figure 10: Results of y and y' (top view).

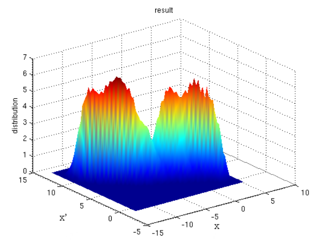

Figure 11: Results of x and x'.

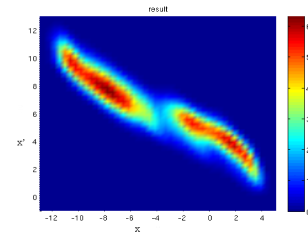

Figure 12: Results of x and x'(top view).

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