GENERAL RESULTS ON THE NATURE OF FEL AMPLIFICATION

Stephen D. Webb, Tech-X Corporation, Boulder, CO 80303, USA*
Vladimir N. Litivnenko, Gang Wang, Brookhaven National Laboratory, Upton, NY 11973, USA

Abstract

Free-electron lasers (FELs) are increasingly important tools for the material and biological sciences, and although numerical and analytical theory is extensive, a fundamental question about the nature of the FEL growing modes has remained unanswered. In this proceeding, we present results of a topological nature concerning the number of amplifying solutions to the 1-dimensional FEL equations as related to the energy distribution of the electron bunches.

INTRODUCTION

Free-electron lasers [1, 2] are a tunable, intense, transversely coherent light source which can provide intense, short pulses of photons into the hard X-ray regime [3]. The FEL process has been explored extensively both theoretically [4, 5, 6] and numerically [7, 8]. These treatments generally regard an energy distribution in the electron bunch which is either monoenergetic (which leads to a cubic equation for the growth rate) or some more elaborate single-peaked distribution such as a Lorentzian or Gaussian distribution. Real LINACs, such as the driver for the LCLS FEL, yield much more complicated energy distributions.

Theoretical descriptions of the free-electron laser process in one dimension lead to a single resonant wavelength for a bunch with average energy γ_0 given by

$$\lambda_r = \frac{\lambda_w}{2\gamma_0^2}\left(1 + \langle a_w^2 \rangle\right) \tag{1}$$

where λ_w is the wiggler period and $a_w = eB_w/k_w mc$ is the wiggler parameter. Resonant interaction between the individual electron trajectories in the wiggler and the laser field leads to exponential growth in the laser field envelope. There is a single mode which is amplified by the FEL at the resonant wavelength under the assumption of a beam with a single energy peak. For the purposes of this discussion, a "peak" refers to any point where the energy distribution is a local maximum. In that context, the energy distribution F has a vanishing derivative, viz.

$$\frac{dF}{d\mathcal{E}} = 0 \tag{2}$$

where $\mathcal{E} = \gamma mc^2$ is the total energy.

The growth rate of this mode is characterized by the FEL dispersion relation

$$s = \hat{D}(s) \tag{3}$$

* swebb@txcorp.com, formerly Brookhaven National Laboratory, Upton, NY 11973, USA.

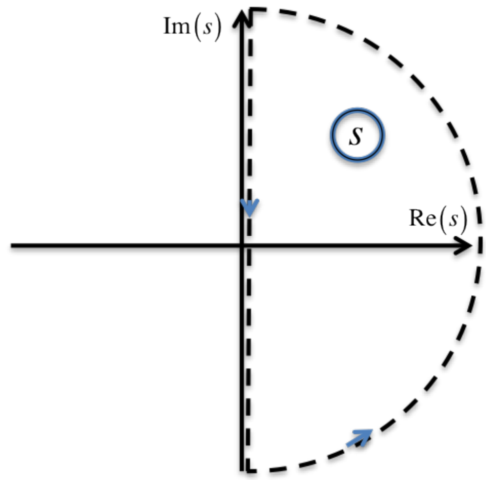

Figure 1: Contour enclosing all the amplifying solutions of a dispersion relation in Laplace space.

where space charge forces have been neglected and

$$\hat{D}(s) = \int d\hat{P} \frac{d\hat{F}}{d\hat{P}} \frac{1}{s + \imath(\hat{\Delta} + \hat{P})} \tag{4}$$

is the dispersion integral. Here, $\hat{F}$ is the normalized energy distribution, $\hat{\Delta}$ is the normalized detuning from resonance[1], $\hat{P}$ is the normalized deviation from some chosen reference energy, and the s_j that solve the dispersion relation appear in the linear solution for the fields

$$E(z) = \sum_j E_j e^{s_j \hat{z}} \tag{5}$$

For $\kappa - N$ distributions, it is known that there are $N + 2$ solutions to the above dispersion relation [9], and thus, in the limit of a Gaussian distribution, there are an infinite number of roots.

Exact analytic expressions for the roots are impossible outside of a monoenergetic or $\kappa - 1$ (Lorentzian) distribution. However, because some of these modes are amplifying and others are not, it is useful to at least quantify how many of these modes are growing, as these are the modes of interest.

NYQUIST DIAGRAMS

Historically, it was found that for linear systems of circuits the region of stability was much larger than a crude

[1] Conventionally, the notation for this is $\hat{C}$, but we opt for a different notation to prevent confusion with the labeling for contours later.

ISBN 978-3-95450-115-1

analysis would have indicated. Herbert Nyquist explained this by considering a contour in frequency space [10] and determining its resulting winding number either analytically, numerically, or experimentally. The original work was done in frequency space, but a Fourier transform can be transformed into a Laplace transform by taking $\omega \mapsto \imath s$, and it is in Laplace space that we consider the FEL amplification process.

From the argument principle of complex analysis[2], it can be shown that the winding number of the above contour when mapped by the dispersion relation is equal to the number of zero minus the number of poles inside that contour.

These Nyquist diagrams allow the measurement of regions of stability, and have been used as a standard tool in electrical engineering, as well as in the discussion of instabilities in plasmas [12], and in analyzing various instabilities and control systems in accelerators (for example [13, 14, 15] among other instances).

FEL GROWING MODES

By using the method of Nyquist diagrams, consider the above dispersion relation with a contour closed in the right half-plane as in Fig. 1. The winding number of this contour is equal to the number of zeros minus the number of poles inside the contour. Thus, if the winding number and number of poles inside the contour is known, the number of modes with Re$(s) > 0$ is given by

$$Z = W + P \tag{6}$$

This corresponds to the number of growing modes.

The details of this calculation are too lengthy for the present proceeding, but can be found in [16]. The essential conclusion is that $D(s)$ vanishes as $|s| \to \infty$, and so the winding number is dictated entirely by the behavior of the contour of $s = \imath t$ for $t = (\infty, -\infty)$, and how many times this wraps around the origin.

As an example of this behavior, consider the energy distribution in Fig. 2 and the corresponding Nyquist diagram in Fig. 3.

Because the detuning is merely a vertical shift of the contour, the Nyquist diagram is universal for a given energy distribution, and the total number of growing modes may be calculated by simply observing the shape of the diagram without concern for the specific detuning parameter.

The Nyquist diagram in Fig. 3, for example, predicts the existence of two amplifying modes at zero detuning. Because the energy distribution was chosen to be a double-Lorentzian distribution, the dispersion relation has a closed polynomial form. The resulting solutions are shown in Fig. 4 where, indeed, at zero detuning there are two growing modes.

In general, the points where the contour crosses the imaginary axis are related to the zeros of the derivative of the normalized energy distribution function, which is

[2] for example, see [11] for a rigorous discussion of this theorem.

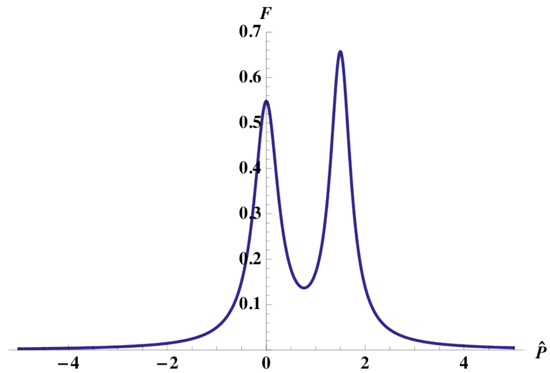

Figure 2: Two peak energy distribution.

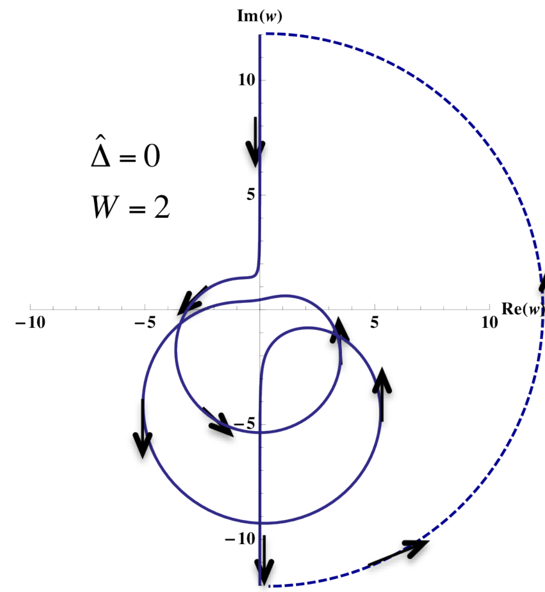

Figure 3: The Nyquist diagram at zero detuning predicts two growing modes.

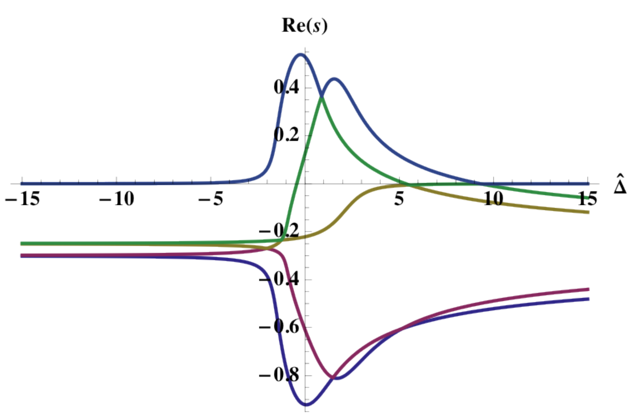

Figure 4: Solutions for the dispersion relation with an energy distribution in Figure 2.

ISBN 978-3-95450-115-1

the number of peaks. Specifically, the contour crosses the imaginary axis when

$$\hat{F}'(\hat{\Delta} - t) = 0 \tag{7}$$

for a given detuning. Thus, in general, for an energy distribution with N local maxima in its energy distribution, one expects at most N amplifying modes. Furthermore, for $N = 1$, an exact form for the short wavelength cutoff is found to be

$$\hat{\Delta}^* = -\int d\hat{P}\, \frac{d\hat{F}}{d\hat{P}} \frac{1}{\hat{P}} \tag{8}$$

and this scales as $\hat{\Delta}^* \sim \sigma^{-2}$, where σ is the normalized energy spread.

DISCUSSION

The resonant wavelength at which a single electron radiates depends upon the single electron's energy. In order to "communicate" with other electrons and, therefore, enter the high gain regime, a large number of electrons must have close to the same energy and be physically close to each other in space. Narrow peaks in the local energy distribution, as those described above, lead to collections of electrons lasing at a similar resonant wavelength. Thus, if there is some local energy distribution within the electron bunch that leads to multiple collections of electrons all radiating within the bandwidth of a single group, but outside the bandwidth of the other groups, then each group with radiate more or less independently of each other. This is the physical origin of the multiple growing modes.

As the energy peaks begin to get close enough together that the electrons can cross-communicate, some constructive interference can begin to form, until the peaks totally overlap and the solutions become degenerate. Thus, if the two-Lorentzian distribution in Fig. 2 were to have their peaks merge closer and closer, eventually the resulting growth would reduce to the problem of a single Lorentzian distribution.

ACKNOWLEDGMENTS

The authors would like to thank Mike Blaskiewicz (BNL) for helpful discussions.This work supported in part by by Brookhaven Science Associates, LLC under Contract No. DE-AC02-98CH10886 with the U.S. Department of Energy.

REFERENCES

[1] J. M. J. Madey, J. App Phys. **42** 1906 (1971).

[2] D. Deacon, L. Alias, J. M. J. Madey, G. Ramian, H. Schwettman, and T. Smith, Phys. Rev. Lett. **38**, 892 (1977).

[3] P. Emma, *et al.*, Nature Photonics **4**, pp. 641-647 (2010).

[4] A. M. Kondratenko and E. L. Saldin, Part. Accel. **10**, 207 (1980).

[5] R. Bonifacio, C. Pellegrini, and L. Narducci, Opt. Commun. **50**, 373 (1984).

[6] E. L. Saldin, E. A. Schneidmiller, and M. V. Yurkov, Nucl. Instrum. and Methods **A313**, 555 (1992).

[7] S. Reiche, Ph.D. thesis, University of Hamburg (1999).

[8] W. Fawley, Tech. Rep., LBNL (1995).

[9] S. D. Webb and V. N. Litvinenko, in *Proc. of FEL'10 Conf.*, 56 (2010).

[10] H. Nyquist, Bell System Technical Journal, **11**, 126 (1932).

[11] M. Ya. Antimirov, A. A. Kolyshkin, and R. Vaillancourt, *Complex Variables*, Academic Press (1998).

[12] O. Penrose, Phys. Fluids **3**, 258 (1960).

[13] R. L. Pease, IEEE Trans. Nucl. Sci. **12**, 3 (1965).

[14] C. J. Bochetta, *et al.*, in *Proc. of PAC '97*, 2 (1997).

[15] E. Vogel, Phys. Rev. ST - Acc. Beams **10**, 052001 (2007).

[16] S. D. Webb, V. N. Litvinenko, and G. Wang, Phys. Rev. ST - Acc. Beams (submitted for publication) (2012).