RADIAL EIGENMODES FOR A TOROIDAL WAVEGUIDE WITH RECTANGULAR CROSS SECTION *

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Abstract

In applying mode expansion to solve the CSR impedance for a section of toroidal vacuum chamber with rectangular cross section, we identify the eigenvalue problem for the radial eigenmodes which is different from that for a cylindrical structure. In this paper, we present the general expressions of the radial eigenmodes, and discuss the properties of the eigenvalues on the basis of the Sturm-Liouville theory.

INTRODUCTION

The studies of coherent synchrotron radiation (CSR) induced impedances or wakefields are important for the generation and transport of high-brightness electron beams in the designs of modern accelerators. These studies require solving the electromagnetic fields generated by a short bunch moving from (to) a straight path to (from) a section of a circular orbit, bounded by straight waveguides connected with a toroidal vaccum chamber. The impedance problem was thoroughly analyzed earlier [1] for a perfect ring orbit in a toroidal waveguide with rectangular cross section by solving the inhomogeneous Helmholtz equations. Recently much progresses has been made in solving the parabolic field equations as an approximation to the Helmholtz equations [2-5]. These approaches are applied to extend the impedance calculation for a circular toroidal waveguide to more general geometries involving sections of toroidal chamber connected to straight waveguides, by mode expansion [2] or by numerically solving the parabolic equations using meshes [5].

In our study, instead of solving the parabolic equations, we carry out mode expansion by directly using eigen-modes of the Helmholtz equation with radial boundary conditions for the section of toroidal waveguide. As the first part of this study, in this paper we investigate the properties of these radial eigenmodes. We will show that in using the method of separation of variables for the homogeneous Helmholz equation, one encounters the situation where the eigenvalue problem for the Bessel equation associated with the radial variable is different from that derived for a cylindrical structure. In the following sections, we will develop basic equations and identify the eigenvalue problem for the radial mode. We will then present the general expression for the eigenfunctions, and discuss the properties of the eigenvalues based on the Sturm-Liouvilles theory for self-adjoint ODE. An approximate analytical expression for the eigenvalues is also presented.

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BASIC EQUATIONS

Following the analysis in Ref. [1], we need to solve the inhomogeneous wave equations for E_z and H_z in the toroidal waveguide,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right)+\frac{R^2}{r^2}\frac{\partial^2 E_z}{\partial s^2}+\frac{\partial^2 E_z}{\partial z^2}-\frac{\partial^2 E_z}{c^2\partial t^2}=\frac{1}{\epsilon_0}\frac{\partial\rho}{\partial z}, \tag{1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_z}{\partial r}\right)+\frac{R^2}{r^2}\frac{\partial^2 H_z}{\partial s^2}+\frac{\partial^2 H_z}{\partial z^2}-\frac{\partial^2 H_z}{c^2\partial t^2}=-\frac{1}{r}\frac{\partial(J_s r)}{\partial z}. \tag{2}$$

The section of toroidal waveguide is bounded by the upper and lower walls at $z=\pm g$ as well as by the inner and outer walls with radius $r=a$ and $r=b$. The design orbit for the electron beam inside the toroidal chamber has radius R, and s is the longitudinal coordinate. The boundary conditions on the perfect conducting waveguide walls are

$$E_z|_{r=a}=E_z|_{r=b}=0,\quad \left.\frac{\partial E_z}{\partial z}\right|_{z=g}=\left.\frac{\partial E_z}{\partial z}\right|_{z=-g}=0,$$

$$\left.\frac{\partial H_z}{\partial r}\right|_{r=a}=\left.\frac{\partial H_z}{\partial r}\right|_{r=b}=0,\ H_z|_{z=g}=H_z|_{z=-g}=0,$$

The other field components, (E_r, E_s, H_r, H_s), can be obtained in terms of E_z and H_z. The above wave equations can be solved by applying Fourier expansions

$$E_z=\int d\omega\, e^{-i\omega t}\sum_{l=0}^{\infty}\cos\alpha_l(z+g)\,E_{zl}(r,s;\omega),$$

$$H_z=\int d\omega\, e^{-i\omega t}\sum_{l=0}^{\infty}\sin\alpha_l(z+g)\,H_{zl}(r,s;\omega)$$

to Eqs. (1) and (2), which incorporate boundary conditions at $z=\pm g$ by setting $\alpha_l=l\pi/(2g)$. This gives rise to inhomogeneous Helmholtz equations for $E_{zl}(r,s;\omega)$ and $H_{zl}(r,s;\omega)$, the solutions of which can be written as the sum of particular solutions involving source terms and solutions for the homogeneous Helmholtz equation. In this study we are particularly interested in the homogeneous Helmholtz equation, which has the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W_l}{\partial r}\right)+\frac{R^2}{r^2}\frac{\partial^2 W_l}{\partial s^2}+\gamma_l^2 W_l=0 \tag{3}$$

for $\gamma_l^2=(\omega/c)^2-\alpha_l^2$ and $W_l(r,s)=E_{zl}(r,s;\omega)$ or $H_{zl}(r,s;\omega)$.

For a circular toroidal waveguide [1], W_l satisfies the periodic condition $W_l(r,s)=W_l(r,s+C)$ as well as the boundary conditions

$$W_l|_{r=a} = W_l|_{r=b} = 0, \qquad (\text{for } W_l = E_{zl}), \tag{4}$$

$$\left.\frac{\partial W_l}{\partial r}\right|_{r=a} = \left.\frac{\partial W_l}{\partial r}\right|_{r=b} = 0 \qquad (\text{for } W_l = H_{zl}). \tag{5}$$

The periodic condition in s allows one to expand W_l in Eq. (3)

$$W_l(r,s) = \sum_{n=-\infty}^{\infty} e^{i2\pi ns/C}\, W_{ln}(r),$$

where $W_{ln}(r)$ is the solution of the Bessel equation

$$\frac{d}{dr}\left(r\frac{dW_{ln}}{dr}\right) + \left(\gamma_l^2 r - \frac{n^2}{r}\right) W_{ln} = 0. \tag{6}$$

With the cross product of Bessel functions

$$p_\mu(\gamma_l a, \gamma_l r) = J_\mu(\gamma_l a)Y_\mu(\gamma_l r) - Y_\mu(\gamma_l a)J_\mu(\gamma_l r), \tag{7}$$
$$q_\nu(\gamma_l a, \gamma_l r) = J'_\nu(\gamma_l a)Y_\nu(\gamma_l r) - Y'_\nu(\gamma_l a)J_\nu(\gamma_l r), \tag{8}$$
$$s_\nu(\gamma_l a, \gamma_l r) = J'_\nu(\gamma_l a)Y'_\nu(\gamma_l r) - Y'_\nu(\gamma_l a)J'_\nu(\gamma_l r) \tag{9}$$

and the synchronous condition $\omega = \frac{\beta n}{R}$, for $\beta = v/c$ representing the velocity of the electron beam, one finds resonance conditions for n when

$$W_{ln}(b) = p_n(\gamma_l a, \gamma_l b) = 0 \qquad (\text{for } W_l = E_{zl}),$$
$$W'_{ln}(b) = s_n(\gamma_l a, \gamma_l b) = 0 \qquad (\text{for } W_l = H_{zl}).$$

For general geometries involving a section of toroidal waveguide connected to straight waveguides, the periodic condition in s no longer apply. We can then solve the problem by mode expansion for each longitudinal section of waveguide and matching fields at the interfaces. The main focus of this paper is to understand the properties of the radial eigenmodes for a section of toroidal waveguide with arbitrary longitudinal range $s_1 < s < s_2$, as needed for the follow-up mode expansion studies.

With separation of variables $W(r,s) = U(r)V(s)$, and for $k^2 = \gamma_l^2$, one can write Eq. (3) as

$$r^2\left(\frac{1}{U}\frac{d}{rdr}r\frac{dU}{dr} + k^2\right) = -R^2\frac{d^2V}{Vds^2} = \chi.$$

This results in a Bessel equation for $U(r)$

$$\frac{d}{dr}r\frac{dU}{dr} + (k^2 r - \chi/r)U = 0 \tag{10}$$

and an oscillator equation for $V(s)$

$$d^2V/ds^2 + \frac{\chi}{R}V = 0\ .$$

Here for the general discussion of the properties of Eq. (3), we omit the label l for W, U, V and χ. The constants $\chi = \chi_i\ (i \geq 1)$ are the eigenvalues of Eq. (10), as a result of the boundary conditions of $U(r)$ at $r = a$ and $r = b$:

$$U(a) = U(b) = 0 \qquad (\text{for } W = E_{zl}), \tag{11}$$
$$U'(a) = U'(b) = 0 \qquad (\text{for } W = H_{zl}). \tag{12}$$

One can show that the solution for Eqs. (10) and (11) is

$$U_i(r) = p_{\mu_i}(ka, kr) \tag{13}$$

with the eigenvalues $\chi_i = \mu_i^2$ such that $p_{\mu_i}(ka, kb)$=0. Similarly the solution for Eqs. (10) and (12) is

$$U_i(r) = q_{\nu_i}(ka, kr) \tag{14}$$

with the eigenvalues $\chi_i = \nu_i^2$ such that $s_{\nu_i}(ka, kb)$=0. The general solution of $W = E_{zl}(r, s; \omega)$ for $s_1 < s < s_2$, which satisfies Eqs. (3) and (4), is

$$W(r,s) = \sum_{i=1}^{\infty} p_{\mu_i}(ka, kr)\ \left(A_i \cos\frac{\mu_i s}{R} + B_i \sin\frac{\mu_i s}{R}\right),$$

and the general solution for $W = H_{zl}(r, s; \omega)$, which satisfies Eqs. (3) and (5), is

$$W(r,s) = \sum_{i=1}^{\infty} q_{\nu_i}(ka, kr)\ \left(C_i \cos\frac{\nu_i s}{R} + D_i \sin\frac{\nu_i s}{R}\right).$$

Note that unlike the usual eigenvalue problem for a cylindrical structure, such as a pillbox cavity, where the orders of Bessel functions are given while the eigenvalues reside in the arguments of the Bessel function, here for the radial eigenmodes in a section of toroidal waveguide, things are twisted around in Eqs. (13) and (14), where the arguments of the Bessel function are given while the eigenvalues reside in the order of the Bessel functions.

DISCUSSION OF THE EIGENVALUE PROBLEM

Since our new eigenvalue problem for the Bessel equation (Eqs. (10)-(12)) is different from that in most applications of Bessel equations, we need to go back to the fundamental theory on the Sturm-Liouville equation and demonstrate where the difference takes place and what is the implication of these differences to our eigenvalue properties.

Recall that the Sturm-Liouville equation takes the form

$$\mathcal{L}[y] := \left(\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)\right)y = -\lambda w(x)y \tag{15}$$

for x over a finite interval (a, b). Here q, w, p and p' are real continuous functions, with p and w being positive definite on the interval. The boundary conditions are

$$\alpha_1 y'(a) + \alpha_2 y(a) = 0,\ \ \beta_1 y'(b) + \beta_2 y(b) = 0 \tag{16}$$

for arbitrary real constants $\alpha_1, \alpha_2, \beta_1$ and β_2. The Sturm-Liouville theory [6] proves that the eigenvalues λ_i are a set of real discrete numbers which has a lower limit and increases to infinity without bound,

$$\lambda_1 < \lambda_2 < \cdots < \infty . \quad (17)$$

The corresponding eigenfunctions are oscillatory functions which are orthorgonoal to each other, and form a complete set in the space of piecewise smooth functions on the interval (a, b). The oscillation is more rapid for larger eigenvalues. The orthogonality of the eigenmodes associated with eigenvalues λ_i and λ_j is given by

$$\int_a^b y_i(x) y_j(x)\, w(x) = N_i \delta_{ij} , \quad (18)$$

for the normalization factor N_i. The Rayleigh quotient

$$\lambda_i = \frac{\left[-p\, y_i y_i'\right]_a^b + \int_a^b \left[p\, y_i'^2 - q y_i^2\right] dx}{\int_a^b y_i^2 w dx} \quad (19)$$

ensures that all λ_i are positive definite when $q(x) \leq 0$ as well as when the boundary conditions have the forms of Eqs. (11) or (12).

The usual eigenvalue problem for the Bessel equation

$$\frac{d}{dr}\left(r \frac{dy}{dr}\right) + \left(k^2 r - \frac{n^2}{r}\right) y = 0,$$

with boundary conditions Eqs. (11) or (12), is often derived from Helmholtz equation for a cylindrical structure. It has the Sturm-Liouville coefficient functions

$$p(r) = r,\ w(r) = r,\ q(r) = -\frac{n^2}{r},\ \lambda = k^2 . \quad (20)$$

For a given n, one has $y(r) = C J_n(kr) + D Y_n(kr)$. The task is to find $\lambda_i = k_i^2$ and the coefficients C and D so as to meet the boundary conditions at $r = a$ and $r = b$.

In contrast, our new eigenvalue problem for a toroidal waveguide is given by the boundary conditions Eqs. (11) or (12) for Eq. (10), or,

$$\frac{d}{dr} r \frac{dy}{dr} + (k^2 r + \lambda / r) y = 0. \quad (21)$$

The coefficient functions are

$$p(r) = r, w(r) = \frac{1}{r}, q(r) = k^2 r, \lambda = -\chi . \quad (22)$$

Note that $q(r)$ and $w(r)$ in Eq. (22) switch role from those in Eq. (20), and in addition we now have $q(r) > 0$ for the case $k^2 > 0$. Consequently, we find that the Sturm-Liouville theory on the real and discrete aspects of eigenvalues and on the orthogonal and complete aspects for eigenfunctions still hold for the toroidal waveguide case. However, from the Rayleigh quotient in Eq. (19), the possibility of $q > 0$ in Eq. (22) no longer guarantees the positiveness of the eigenvalues. This is consistent with the observation that when we substitute $\lambda = -\chi$ to Eq. (17), we get

$$-\infty < \cdots < \chi_2 < \chi_1 \quad (23)$$

where the upper bound χ_1 can be either positive or negative, depending on the value of k^2. Also unlike the case for a cylindrical structure where the weight function $w(r) = r$ is used for the orthogonality condition in Eq. (18), the weight function for the toroidal waveguide becomes $w(r) = 1/r$ instead.

Remark that when $\chi_i = \mu_i^2 < 0$, the orders of Bessel functions in Eqs. (7)-(9) are purely imaginary. Nonetheless, as we have proved, the eigenfunctions in Eqs. (13) and (14) are real functions of r. For $(b - a) \ll R$, it can be shown that the Bessel functions, with real or imaginary order, can be well approximated by the dominant terms of the Olver expansions [7] in terms of the Airy functions. Under this approximation, the eigenvalues in Eqs. (13) and (14) are nearly the same, $\mu_i \approx \nu_i$ for all $i \geq 1$, and

$$\chi_i = \mu_i^2 \approx \left[k^2 - \left(\frac{i\pi}{b-a}\right)^2\right] a^2. \quad (24)$$

Here despite the similarity of the radial eigenvalues in Eq. (24) with those for a straight rectangular waveguide, the radial eigenfunctions behave very differently from the sinusoidal eigenfunctions in the latter case, especially for those with lower mode indices. More discussions on the asymptotic expressions of μ_i can be found in Ref. [9].

For some example geometries, our studies show that the eigenvalues in Eq. (24), as well as the eigenfunctions for the radial eigenmodes in Eqs. (13) and (14), agree very well with direct numerical solutions of Eqs. (10)-(12). These numerical solutions are obtained using the ODE solver extended from the one provided by Warnock [8]. The results obtained in this paper can be further applied to study the resonance behaviour of the CSR impedance for general geometries involving sections of toroidal waveguide.

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