NUMERICAL APPROACHES FOR SIMULATION OF STOCHASTIC COOLING IN 2D PHASE SPACE

M. Dolinska, Frankfurt University, Germany, INR Kiev, Ukraine
C. Dimopoulou, A. Dolinskii, F. Nolden, GSI, Darmstadt, D-64291, Germany

Abstract

A consolidated fluid-dynamics algorithm for the analysis of beam dynamics under the influence of the electromagnetic field is presented. Aiming at simulating stochastic cooling of particle beams in 2D space, a numerical algorithm solving the 2D Fokker-Planck Equation is described. Some results of test runs for the simulation of the stochastic cooling in the Collector Ring are discussed.

INTRODUCTION

One important problem in accelerator physics is to study the dynamics of charged particles under the influence of electromagnetic fields and noise.

Stochastic cooling (SC) is a powerful tool for improving beam quality in storage rings. It is also used for accumulation of high intensity ion beams. One of the popular methods to calculate beam dynamics is by solving the Fokker-Planck equation. In ref.[1] the fast and accurate numerical method using a finite difference technique to solve of 1D Fokker-Planck equations describing the stochastic cooling of a beam in a storage ring is presented.

However, for other investigations it is important to consider beam cooling in the transversal and momentum spaces simultaneously, taking into account coupling effects between these spaces. In this work we present one of the different approaches for solution of the 2D FPE using an efficient mathematical solver, which can be applied for time dependent current drive problems.

NUMERICAL SOLUTION OF 2D FPE

The general Fokker-Planck Equation (FPE) for N_v variables $\mathrm{X} = (x_1; x_2; : : : x_N)^{\mathrm{T}}$ has the form [2]

$$\frac{\partial\psi(\mathrm{X},t)}{\partial t} = -\sum_{i=1}^{N_v}\frac{\partial}{\partial x_i}F_i(\mathrm{X})\psi(\mathrm{X},t) + \sum_{i=1}^{N_v}\sum_{j=1}^{N_v}\frac{\partial^2}{\partial x_i\partial x_j}D_{ij}(\mathrm{X},t)\psi(\mathrm{X},t), \quad (1)$$

where F_i is called the drift vector and D_{ij} is the diffusion tensor. These coefficients characterize the given stochastic cooling system and ψ is the 2D Probability Density Function (PDF) describing for example the momentum spread (x_1) and emittance (x_2) of a beam.

We consider a linear 2D FPE, where only two phase coordinates x_1, x_2 (the momentum spread and beam emittance) are taken. The two dimensional FPE can be written by using operator L_{FP} :

$$\frac{\partial\psi(x_1,x_2,t)}{\partial t} = L_{FP}\psi(x_1,x_2,t), \quad (2)$$

$$L_{FP} = \left[-\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)F(x_1,x_2) + \left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+2\frac{\partial^2}{\partial x_1\partial x_2}\right)D(x_1,x_2)\right]$$

The first step in the numerical solution of the Eq.(2) is the spatial discretization by which the transformation of a system of partial differential equations to a system of ordinary differential equations is achieved. As one of the methods in this work a standard Galerkin algorithm is chosen [3].

The main idea of the Galerkin algorithm consists of the following. The relevant portion Ω of the phase space is divided into N elements, each with N_d nodes spanning the domain Ω_e. This method is used to find the probability density function $\psi(x_1,x_2,t) = \hat{\psi}(X,t)$ which is defined by the theorem of total probability, as

$$\hat{\psi}(X,t) = \int_{\Omega^2}\psi(X,t|X_0)\varphi(X_0)dX_0, \quad (3)$$

where Ω is the chosen domain for the function interpolation, φ is the interpolation function. Within eth element the function $\hat{\psi}(X,t)$ is interpolated according to the scheme

$$\hat{\psi}\left(x_1^e,x_2^e,t\right) = \sum_{k=1}^{2}\sum_{l=1}^{2} N_{kl}^e\left(x_1^e,x_2^e\right)\psi_{kl}^e\left(x_1^e,x_2^e,t\right), \quad (4)$$

where $N_{kl}^e(\mathbf{X})$ are the element shape functions, the ψ_{kl}^e are the values of the probability density function at the nodes. Further adaptations in the meaning of the Galerkin method should be applied primarily to the differential part. Globally assembling all elements, a set of N equations can be constructed, which in matrix form is given by

$$U\frac{\partial\psi}{\partial t} + K\psi = 0, \quad (5)$$

with initial condition $\psi(0)=\psi$, where ψ is a vector containing the values of the joint probability density function at the nodal points. The matrices U and K are defined by shape functions $N_{kl}^e(\mathbf{X})$ [3].

Having addressed the issue of spatial discretization, the next step is temporal discretization, that is converting the system of first-order ODE's to a set of linear algebraic equations. So the Eq.(5) is further discretized in time using the Crank–Nicholson scheme, which is described in detail in ref.[1] for 1D FPE. The time discretization gives the following recurrence relation:

$$\left[U + \Delta t\cdot\theta\cdot K\right]\psi_i^{k+1} = \left[U - \Delta t\cdot(1-\theta)\cdot K\right]\psi_i^k, \quad (6)$$

here the parameter θ can be chosen in the range of 0-1, Δt is the time step. The Crank–Nicholson scheme is applied with $\theta = 0.5$ for stability. Eq.(6) can be written by a more simplified form

$$G\psi_i^{k+1} = R, \quad (7)$$

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where G=[U+Δtθ K] is the matrix with N^2xN^2 elements and R=[U-Δt(1-θ)K]ψ_i^k is the vector of N^2 elements. After simple mathematical treatments the G matrix and R vector can be defined. One can see that G is the nine-diagonal matrix and the elements of R matrix have the following form:

$$r_{i,j} = \psi^k_{i,j} + W\cdot\left(-\frac{F_{i+1,j}\psi^k_{i+1,j} - F_{i-1,j}\psi^k_{i-1,j}}{2\Delta x_1} - \frac{F_{i,j+1}\psi^k_{i,j+1} - F_{i,j-1}\psi^k_{i,j-1}}{2\Delta x_2} + \right.$$
$$+\frac{D_{i+1,j}\psi^k_{i+1,j} - 2D_{i,j}\psi^k_{i,j} + D_{i-1,j}\psi^k_{i-1,j}}{\Delta x_1^2} + \frac{D_{i,j+1}\psi^k_{i,j+1} - 2D_{i,j}\psi^k_{i,j} + D_{i,j-1}\psi^k_{i,j-1}}{\Delta x_2^2} +$$
$$\left.+\frac{D_{i+1,j+1}\psi^k_{i+1,j+1} - D_{i+1,j-1}\psi^k_{i+1,j-1}}{\Delta x_1\Delta x_2} - \frac{D_{i-1,j+1}\psi^k_{i-1,j+1} - D_{i-1,j-1}\psi^k_{i-1,j-1}}{\Delta x_1\Delta x_2}\right)$$

The matrix form Eq.7 represents a system of N^2 linear algebraic equations which must be solved at each time step to advance the numerical solution for the density function to stationary. The final computational issue is that one needs a fast numerical solver of Eq.7. Direct methods such as Gauss elimination or decomposition become prohibitively expensive. However, due to the particular nine-diagonal form of the G matrix, one can use an iterative technique. This gives advantages with respect of storage and, hence, efficiency on high performance computing platforms. Only the action of the coefficient G matrix and the R vector is computed. This reduces the memory requirements of the algorithm to those of the iterative solver itself, associated with the generation of the basis vectors in the subspace used by the solver. In terms of required resources, the direct solution of a 2D FPE requires gigabytes of memory, while the iterative method used in this algorithm requires less then one gigabyte, well within the present capabilities of today's computing platforms.

Implementation of an iterative method requires to save the G matrix elements by a special format. Let us assign to the G global matrix elements three indexes *a, i* and *j*, which indicate the following ranges: *a = 1 … 9* – number of the diagonal; *i = 1 … N* – number of row of the local matrix, *j = 1 … N* – number of row of the local matrixes in the global matrix. The $g_{a,i,j}$ matrix elements are obtained by simple calculations and one gets for global matrix elements $g_{a,i,j}$:

$$g_{1,i,j} = -\frac{T}{4}\frac{D_{i-1,j-1}}{\Delta x_1\Delta x_2};\quad g_{2,i,j} = -\frac{T}{2}\left(\frac{2D_{i,j-1}}{\Delta x_2^2} + \frac{F_{i,j-1}}{\Delta x_2}\right);\quad g_{3,i,j} = \frac{T}{4}\frac{D_{i+1,j-1}}{\Delta x_1\Delta x_2}$$
$$g_{4,i,j} = -\frac{T}{2}\left(\frac{2D_{i-1,j}}{\Delta x_1^2} + \frac{F_{i-1,j}}{\Delta x_1}\right);\quad g_{5,i,j} = 1 + 2\cdot\left(\frac{D_{i,j}}{\Delta x_1^2} + \frac{D_{i,j}}{\Delta x_2^2}\right);$$
$$g_{6,i,j} = -\frac{T}{2}\left(\frac{2D_{i+1,j}}{\Delta x_2^2} - \frac{F_{i+1,j}}{\Delta x_1}\right);\quad g_{7,i,j} = \frac{T}{4}\frac{D_{i-1,j+1}}{\Delta x_1\Delta x_2};$$
$$g_{8,i,j} = -\frac{T}{2}\left(\frac{2D_{i,j+1}}{\Delta x_2^2} - \frac{F_{i,j+1}}{\Delta x_2}\right);\quad g_{9,i,j} = -\frac{T}{4}\frac{D_{i+1,j+1}}{\Delta x_1\Delta x_2}.$$

As the method of numerical integration of the system of ordinary linear algebraic equations proved the best the procedure of relaxation method based on a method of successive over-relaxation [4] the vector ψ^{k+1} is calculated. In this method the new calculated component ψ^{k+1} is written into the vector ψ^k directly during the iteration process:

$$\psi^{k+1}_{i,j} = \psi^k_{i,j} + \frac{\omega}{g_{5,i,j}}\left(r_{i,j} - g_{1,i,j}\psi^k_{i-1,j-1} - g_{2,i,j}\psi^k_{i,j-1} - \right.$$
$$g_{3,i,j}\psi^k_{i+1,j+1} - g_{4,i,j}\psi^k_{i-1,j} - g_{5,i,j}\psi^k_{i,j} - g_{6,i,j}\psi^k_{i-1,j+1} -$$
$$\left.g_{7,i,j}\psi^k_{i,j+1} - g_{8,i,j}\psi^k_{i,j+1} - g_{9,i,j}\psi^k_{i+1,j+1}\right)$$

The ω is the parameter, which is close to 1.

TEST RUNS PRELIMINARY RESULTS

To test the proposed solver the "2D FOPLEG" code has been written, where Galerkin method method is implemented. Preliminary results of the test runs are demonstrated in this chapter. For an indication of the solution stability one calculates the PDF evolution over the long cooling time (about 20 sec), which is sufficient to reach the beam equilibrium and to study the solver behaviour. In order to evaluate the accuracy of each numerical method one calculates the relative changes of the 2D PDF volume during the cooling time by formula:

$$\delta(t) = \frac{V(t)}{V(0)} - 1, \qquad (8)$$

where V(t) is the volume of the 2D PDF depending on time. V(0) is the volume of the initial PDF function.

Test Run I

In this test run the initial symmetric 2D PDF is selected as a Gaussian shape and calculated by formula:

$$\psi(x_1, x_2, 0) = \exp\left(-\frac{(x_1 - x_{01})^2}{\sigma_1^2}\right)\exp\left(-\frac{(x_2 - x_{02})^2}{\sigma_2^2}\right) \qquad (9)$$

The approximation (8) admits that the PDF has a center at the point (x_{01} , x_{02}). The value $\psi(x_1, x_2, t) = 0$ on the domain boundary can be used as the boundary condition for every *t*.

The drift *F* and diffusion *D* coefficients for momentum cooling are calculated according the formulae given in ref. [1] and [5], they are assumed to be constant or, by other words, they have no dependency on the time and the PDF shape (the parameters of the SC system needed for *F* and *D* are given in ref. [1]. The *F* and *D* coefficients are taken to be identical in x_1 and x_2 directions.

In this test run one has so called 2D momentum cooling, which is not realistic from the practical point of view. It is only the mathematical test of solvers. There is no coupling between x_1 and x_2 directions. In Fig.1 the 2D PDFs at *t=0* and time at *t=5 s* are shown.

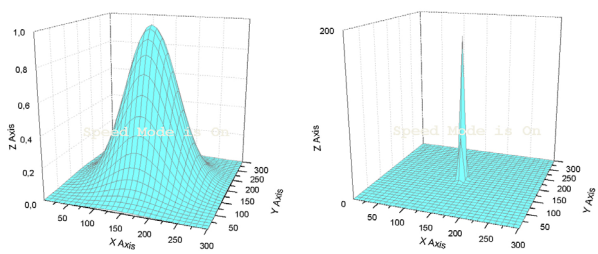

Figure 1: 2D symmetric PDF at t=0 and at t=5 s.

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Test Run II

In this test run the non-symmetrical PDF is calculated by the formula

$$\psi(x_1,x_2,0) = \exp\left(-\frac{(x_1-x_{01})^2}{\sigma_1^2}\right)\exp\left(-\frac{x_2^2}{\sigma_2^2}\right) \qquad (10)$$

where the variable x_1 represents the momentum deviation $\Delta p/p$, and x_2 corresponds to the beam emmitance. For calculation it is used such parameters: $\sigma_1=(\Delta p/p)_{rms,0}=$ *2.3x10^{-3}* and $\sigma_2=\varepsilon_{rms,0}=$ *43 mm mrad*. The range of x_1 is *[-7 x 10^{-3}...+7 x 10^{-3}]* and x_2 is *[-240...+240] mm mrad*. The number of grid points in the 2D space *NxN=500x500*. The coefficients F_{x1} , D_{x1} for momentum cooling are the same as in the Test Run I. For the transversal plane the coefficients F_{x2} , D_{x2} are calculated by well known formula for transverse cooling [5]. We perform the evolution with *10^4* iterations with a time step of $\Delta t = 10^{-4}$.

To be sure that cooling is proceeding correctly we make two calculations of cooling in each direction separately independently from each other. In Fig. 2 the evolution of the rms momentum spread is shown, while the rms value of the emittance is constant over all cooling time. This indicates that there are no mathematical (or numerical) coupling effects between the two spaces. In Fig. 5 the shape of the 2D PDF (left side) is shown in case of momentum cooling only. The volume change of the PDF or the accuracy of the numerical method calculated by Eq. (8) is shown in Fig. 3. One can see that δ(t) is better than 5x10^{-4} and is stable after 2 sec of the cooling time.

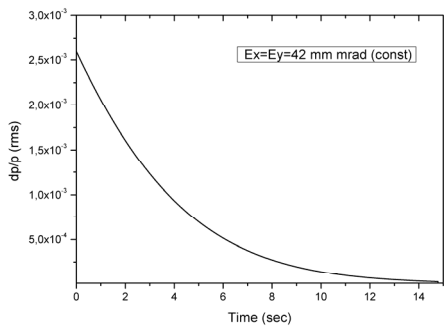

Figure 2: Evolution of the momentum spread. (F_{emit} = D_{emit} = 0).

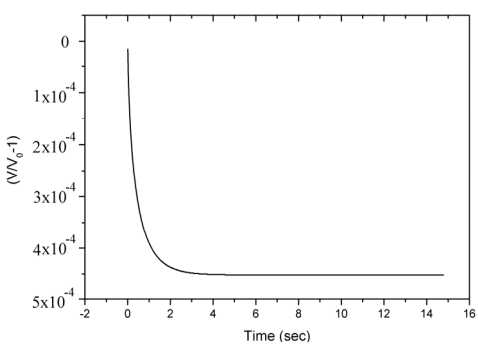

Figure 3: Accuracy of the calculation.

In the next simulation the following conditions are considered: $F_{x1}=D_{x1}=0$ and F_{x2} *and* D_{x2} are not zero. The stochastic cooling system parameters are assumed to be different for transverse cooling than momentum cooling. This means cooling is performed only in the transversal plane. In Fig. 4 the evolution of the rms emittance is shown and in Fig. 5 (right side) the 2D PDF is shown after 1 s of cooling.

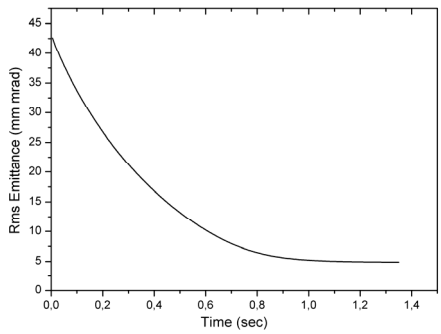

Figure 4: The emittance evolution.

A typical solution run (10^4 time steps) takes around 2-180 min of computational time depending on the mesh size N, which in our simulation is varies in the range 200-500.

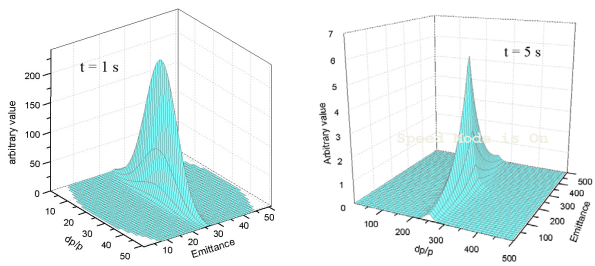

Figure 5: 2D Probability density function indicates: the transversal cooling at time 1 s (left) and the momentum cooling at time of 5 s (right).

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