TESTING OF SYMPLECTIC INTEGRATOR OF SPIN-ORBIT MOTION BASED ON MATRIX FORMALISM

A. Ivanov*, S. Andrianov, N. Kulabukhova, St. Petersburg State University, Russia
R. Maier, Yu. Senichev, D. Zyuzin, FZJ, Julich

Abstract

Investigation of spin-orbit motion in electromagnetic fields requires different numerical methods. Approaches for long-term evolution modelling need both performance and symplecticity. In this paper we discuss matrix maps method for numerical simulation. We examine symplecticity of maps by two ways. First of all, the direct condition of symplecticity is considered. Second approach is an order-by-order symplectification of trancated series map. In the research we examine symplectification and errors in terms of electostatic storage ring.

INTRODUCTION

The most part of beam physics problems can be described using the Hamiltonian presentation. Main properties of similar systems are in common practice of qualitative investigations. But the practical calculations based only on numerical algorithms do not guarantee in general the symplecticity property which is inherent to all Hamiltonian systems. Failing of this property can produce losing of real effects and to acquisition of false effects. That is why all commonly used numerical methods should have the symplecticity property. In that case the simulation process will guarantee adequate and accurate results. The successful results have been achieved in computer modeling of long beam evolution using the Lie algebraic methods. According to this approach a simulator constructs high-order maps and use them for the design, optimization and operation of beamlines. However the practical realization of this powerful approach does not guarantee the symplecticity property automatically. Indeed the realization of Lie methods usually uses truncated series in different forms. As it is known these truncated series have not properties intrinsical to the starting map. There are some works (see, e.g. [1–3]) where some symplectification algorithms are described. These algorithms have numerical character and so have all imperfections residing to all numerical methods and algorithms.

In the report [4] an approach for step-by-step symplectification for the Lie algebraic methods are suggested. In contrast to usual numerical approaches the way uses matrix formalism for the Lie algebraic tools in symbolic mode. In this research we use numerical implementation of matrix formalism. This allows us to create very simple correction via classical optimization procedure, which guarantee the symplecticity properties in all orders up to some hand-picked approximation order.

* 05x.andrey@gmail.com

In the article we will consider the given approach applied to the investigation of spin-orbital motion in an electrostatic storage ring. Particle dynamics is considered in 8-dimensional space. A state of dynamic system is described as $(x,\, x',\, y,\, y',\, S_x,\, S_y,\, \delta E,\, t)$ vector, where x, x' and y, y' are transverse and vertical displacement and velocity respectively; $S_x,\, S_y$ are components of spin vector, t is time variable. Description of orbital motion is based on Newton-Lorentz equation. Spin dynamics is described by BMT-equation. Despite the fact that the symplectic condition are also can be applied for spin map (means skew-symmetric matrix), we will focus on orbital part of maps. The spin normalistaion is controlled by replacing $S_s = \sqrt{1 - S_x^2 - S_y^2}$ in BMT-equation.

MATRIX MAP

Let's introduce a nonlinear system of ordinary differential equations

$$\frac{d}{dt}X = F(t, X). \tag{1}$$

Under the assumptions of $F(0, X_0) = 0$ the system (1) can be presented in the following form

$$\frac{d}{dt}X = \sum_{k=0}^{\infty} P^{1k}(t)X^{[k]}, \tag{2}$$

where $X^{[k]}$ is kronecker power of vector X, matrices P^{1k} can be calculate as

$$P^{1k}(t) = \frac{1}{(k)!}\frac{\partial^k F(t, X_0)}{\partial (X^{[k]})^T}, \quad k = 1, 2, \ldots$$

Note that vector X is equal to $(x_1^{k_1}, \ldots, x_n^{k_n})$, where x_i means ith component of state, $(k)! = k_1! \ldots k_n!$

Solution of system (2) can be written in form

$$X = \sum_{k=0}^{\infty} R^{1k}(t)X_0^{[k]}. \tag{3}$$

Elements of matrices R^{1k} are depended on t and can be calculated in symbolic mode. But such algorithm are quite complex. In this paper we use a numerical implementaton of it. In this case matrices R^{1k} are evaluate in the specific time. After differentiating the equation (3) and taking into account (2) we get

$$\frac{dX}{dt} = \sum_{k=0}^{\infty} \frac{dR^{1k}(t)}{dt}X_0^{[k]},$$

$$\sum_{k=1}^{\infty} \frac{dR^{1k}(t)}{dt}X_0^{[k]} = \sum_{k=1}^{\infty} P^{1k}(t)X^{[k]}.$$

The partial derivatives of this equations with respect to $X_0^{[j]}$ are equal to

$$\frac{dR^{10}(t)}{dt} = \sum_{k=1}^{\infty} P^{1k}(t)(R^{1k})^{[k]},$$
$$\frac{dR^{1j}(t)}{dt} = \sum_{k=1}^{\infty} P^{1k}(t)\frac{\partial X^{[k]}}{\partial (X_0^{[j]})^T}, \quad k = 1, 2, \ldots \tag{4}$$

and define the system of ordinary differential equations. Solution of this system is deterined by matrices R^{1k}. Equations (4) describes the solution of an ODE in general form. Commonly in beam dynamics the motion is described in deviations from reference orbit. In this case map R^{10} is equal to zero.

For integration of equations (4) numerical approach can be used. Note that step-by-step integration method is used only for map building. After that the solution can be calculate with the map (3).

SYMPLECTICITY

The relation (3) can be presented as map transformation

$$X = R \circ X_0. \tag{5}$$

This map R is symplectic if

$$M^* J M = J, \forall X_0, \tag{6}$$

where $M = \partial X / \partial X_0$ and M^* is the transpose of M, E is identity matrix,

$$J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}. \tag{7}$$

As we say above the system of equations (4) can be solved by various numerical methods. After the map is calculated it is possible to compute an error of symplecticity violation directly following the equation (6). The table 1 shows an error analysis for a set of numerical methods and steps of integration, that was used for map computing. The error is calculated as a norm of matrix $\|M^* J M - J\|$.

Table 1: Symplecticity violation

Method/step	$h = 0.2L$	$h = 0.1L$	$h = 0.01L$
Euler method	0.2233	0.1065	0.0104
Runge–Kutta 4th	0.0717	0.0205	0.0119
Symplectic Runge–Kutta 4th	0.0021	0.0004	0.0004

In the table 1 L means length of an element of lattice, h is integration step. In the research we use symplectic 2 stage Runge-Kutta scheme of 4 order [5] that can be described by the scheme (see Table 2).

Table 2: 2-stage 4-order implicit Runge-Kutta scheme

$b_1 + \tilde{c}_1$	$b_1/2$	$b_1/2 + \tilde{c}_1$
$b_1 - \tilde{c}_1$	$b_1/2 - \tilde{c}_1$	$b_1/2$

$$b_1 = 1/2, 2b_1\tilde{c}_1{}^2 = 1/12$$

On the other hand, the relation (6) in case of numerical matrices R^{1k} leads to a system of equations

$$A_0 + A_1\mathbf{X_0}^{[1]} + \cdot + A_k\mathbf{X_0}^{[k]} = 0,$$

where A_i is a numerical vector. Note that this equation must be satisfied for any X_0. It means that the coefficients of each polynom are equal to zero and in this way appropriate corrections of the elements of the matrices R^{1k} can be found. So one can calculate the relations between matrix elemnts and correct them according to it. We have developed a simple tool (see Fig. 1) that allows us to obtain necessary equations. When the equations are quite difficult to solve the standart optimisation procedures can be run.

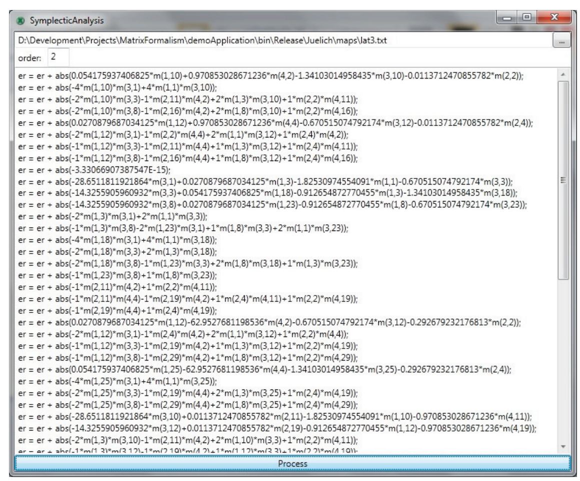

Figure 1: Symplectic condition generation.

One can build a numerical map with the given error (see Table. 1) and order of non-linearity. The difficulties can arise in map concatenation. Obviously, the concatenation of two maps of order k produces map of order k^2. As we expect the same order k of the output map, we neglect of the high-order terms in the map. This immediately leads to symplecticity violation.

The aim of order-by-order symplectification is introduce small correction on the map elements to restore the symplectic property. The given problem has a particular importance in case of reference and design orbits displacement. In particle dynamics it can arrise in fringe fields modelling. The next session is described the result of fringe field modelling in electrostatic storage ring.

ISBN 978-3-95450-122-9

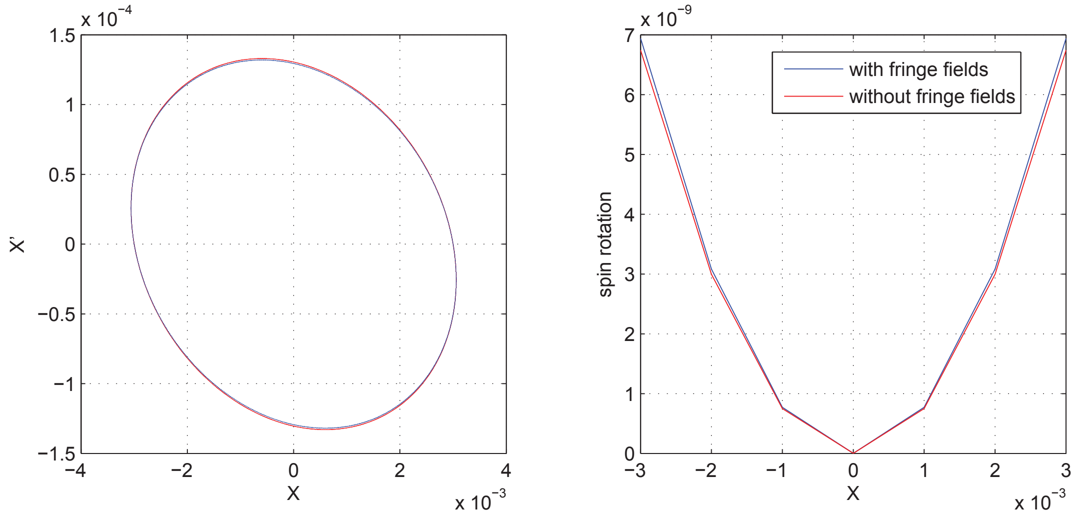

Figure 2: Fringe field modelling (left picture – transverse plane x-x', right picture – spin rotation)

FRINGE FIELDS MODELING

The fringe fields in general case introduces the shift of reference orbit with respect to the design orbit. In map representation of the solution it means that the zero-order part R^{10} is not equal to zero. This entails difficulties with reference orbit investigation.

In the research fringe fields in deflectors have been examined. To achieve the same rotation angle the field in deflector was modified. By a numerical methods of optimization one can find the voltage in a tip of deflector for predefind fringe field configuration and rotation angle. This can provide map $\|R^{10}\| < 10^{-9}$, so we can neglect this term of map. The small errors that are introduced by this neglection can be corrected by order-by-order symplectification.

As shown on Fig. 2 fringe fields have unessential effects both on orbiatl motion and spin dynamics. The same behavior is observed in simulation via COSY Infinity code.

CONCLUSION

In the research two methods for obtaining a symplectic map have been discussed. First one is building map using a numerical step-by-step integration method. This approach allows to estimate map terms up to the necessary order of non-linearity and on a predetermined level of accuracy. The second one is order-by-order symplectification, that provides conditions on map elements which ensure the symplectic condition.

The given technique was tested on spin-orbit prticles motion simulation in an electrostatic storage ring. Fringe fields modeling requires both accuracy in map building and additional procedure of symplectification. The results are coincide with other numerical simulation tools [6].

ACKNOWLEDGMENT

The authors would like to thank Martin Berz for long discussion of numerical approaches and models, and especially for remarks on order-by-order symplectification techniques.

REFERENCES

[1] P.Channel P., J. Scovel, "Symplectic Integration for Hamiltonian Systems," Nonlinearity, 1990. Vol. 3. P. 231–259.

[2] Feng Kang, Qin Meng-Zhao, "The Symplectic Methods for the Computation of Hamiltonian Equations," Lect. Notes in Mathem, Berlin: Springer, 1987. Vol. 1297. P. 1–37.

[3] J.M. Sanz-Serna, "Runge-Kutta Schemes for Hamiltonian Systems," BIT Comp. Sci. Numerical Math. 1988. Vol. 28. P. 877–883.

[4] S. Andrianov, "Order-by-order symplectification of truncated Lie maps," Proceedings of the PAC 2011, 2011. Vol. 3. P. 1787–1789.

[5] W. Oevel, M. Sofroniou, "Symplectic Runge-Kutta schemes II: classification of symmetric methods," `http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.46.5060`

[6] D. Zyuzin, R. Maier, Yu. Senichev, A. Ivanov, S. Andrianov, M. Berz, "Comparison of Different Numerical Modelling Methods for Beam Dynamics in Electrostatic Rings," Proceedings of IPAC2012, New Orleans, Louisiana, USA, 2012. P. 1335–1337.