ANALYTIC CALCULATION OF ELECTRIC FIELDS OF COHERENT THZ PULSES

Markus Schwarz[*], P. Basler, M. v. Borstel, A.-S. Müller, M. Günther, KIT, Karlsruhe, Germany
M. Schmelling, MPIK, Heidelberg, Germany

Abstract

The coherently emitted electric field pulse of a short electron bunch is obtained by summing the fields of the individual electrons, taking phase differences due to different longitudinal positions into account. For an electron density, this sum becomes an integral over the charge density and frequency spectrum of the emitted radiation, which, however, is difficult to evaluate numerically. In this paper, we present a fast method valid for arbitrary bunch shapes. We also include shielding effects of the beam pipe and consider ultra-short bunches, where the high frequency part of the coherent synchrotron spectrum is cut-off not by the inverse bunch length but by the critical frequency of synchrotron radiation. Our technique is applied to bunches, simulated for the linac-based FLUTE accelerator test facility at KIT.

INTRODUCTION

When electron bunches become short compared to the wavelength of interest the radiation is emitted coherently. Due to the large number of electrons per bunch, coherent radiation can be hundred million times more intense than incoherent radiation. The coherently emitted electric field pulse can be calculated by convoluting the electron bunch shape with the electric field of a single electron. In this paper we present an efficient semi-analytic approach valid for arbitrary bunch shapes and general spectra.

To produce radiation in the THz regime requires bunch lengths of the order of 1 ps or below. FLUTE is compact linear accelerator [1], currently constructed at KIT [2], that will investigate, amongst other things, the creation of THz pulses. We apply our method to calculate the electric field pulse of a simulated 5,6 fs electron bunch.

DERIVATION

The electric field pulse coherently emitted by an electron bunch is obtained by summing over the individual fields of all electrons [3]. Here, we assume that the field of a single electron, as detected by an observer at distance R and time t, is given by a plane wave propagating in the positive z-direction [4]

$$E_\omega(R,t) = \mathrm{Re}\left(E_0(\omega)\, \mathrm{e}^{-\mathrm{i}\omega(t-R/c)-\mathrm{i}\phi}\right), \qquad (1)$$

where c denotes the speed of light. In the following, we set $R = 0$ as it merely leads to a time shift. The details of the radiation mechanism result in a frequency dependence of the spectral amplitude $E_0(\omega)$. In the following, the phase ϕ is assumed to be constant for all electrons and we set $\phi = 0$

---
* Markus.Schwarz@kit.edu

in this paper. In summing the fields of the electrons the positions in the bunch are represented by a phase factor $\mathrm{i}\omega\,\Delta\tau$, where $\Delta\tau$ is the displacement relative to the bunch center. Finally, the electric field in the time domain is obtained by integrating over all frequencies

$$E(t) = N_e\, \mathrm{Re}\left(\int_0^\infty E_0(\omega)\,\tilde\rho(\omega)\,\mathrm{e}^{-\mathrm{i}\omega\,t}\,\mathrm{d}\omega\right). \qquad (2)$$

Here, N_e denotes the electron number and $\tilde\rho(\omega)$ denotes the Fourier transform of the normalized charge distribution. Throughout this paper, a $\sim$ above a function denotes the Fourier transform of the function. The main problem is then to solve

$$\epsilon(t) \equiv \int_0^\infty E_0(\omega)\tilde\rho(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega\,. \qquad (3)$$

In principle, (3) can be computed numerically, but the integral converges poorly due to the oscillating nature of the integrand. In [5], (3) was solved analytically for arbitrary bunch profiles by using cubic spline interpolation of ρ and the low-frequency spectrum of synchrotron radiation for E_0. Here, we extend the method to work for general spectra as well.

The first step is to interpolate the charge distribution as [6]

$$\rho_{\mathrm{interpol}}(t) = \sum_{i=0}^N \rho_i f\left(\frac{t-t_i}{\Delta t}\right). \qquad (4)$$

Here, (t_i, ρ_i) are the data points of the charge distribution with $t_i = t_{\min} + i\Delta t$, $i = 0,\ldots,N$ and $f(x)$ denotes the kernel of the cubic spline interpolation. The kernel $f(x)$ has the property $f(0) = 1$ and equals zero at all other integers. For the following to work, it is important that the t_i are equidistant. Using (4), the Fourier transform can be calculated as

$$\begin{aligned}
\tilde\rho(\omega) &= \int_{-\infty}^\infty \rho_{\mathrm{interpol}}(t)\mathrm{e}^{\mathrm{i}\omega\,t}\,\mathrm{d}t \\
&= \Delta t \sum_{i=0}^N \rho_i \mathrm{e}^{\mathrm{i}\omega t_i}\int_{-\infty}^\infty f(\tau)\mathrm{e}^{\mathrm{i}\omega\Delta t\tau}\,\mathrm{d}\tau \\
&= \Delta t\, \tilde\rho_{\mathrm{dis}}(\omega)\,\tilde f(\omega\Delta t)\,. \qquad (5)
\end{aligned}$$

Notice that the Fourier transform factorizes into a discrete Fourier transform depending only on the data, and the Fourier transform of the kernel. The latter is given by [6]

$$\tilde f(k) = -2\left(6 - \frac{18}{2+\cos k}\right)\frac{1-\cos k}{k^4}\,. \qquad (6)$$

05 Beam Dynamics and Electromagnetic Fields

D06 Code Developments and Simulation Techniques

In the second step, we define $\tilde{\epsilon}(\omega) \equiv E_0(\omega)\,\tilde{\rho}(\omega)$ and $\tilde{\epsilon}_j \equiv \tilde{\epsilon}(\omega_j)$ with $\omega_j \equiv j\,\Delta\omega$, $j = 0,\ldots,M$. The integrand in (3) can thus be approximated as

$$\tilde{\epsilon}_{\text{interpol}}(\omega) = \sum_{j=0}^{M} \tilde{\epsilon}_j\, f\left(\frac{\omega - \omega_j}{\Delta\omega}\right). \qquad (7)$$

After inserting (7) into (3), we can apply the same method as in (5) to calculate $\epsilon(t)$ (with $E_0(\omega) \equiv 0$ for $\omega < 0$ understood, allowing to extend the lower integration limit in (3) to $-\infty$). The result simply reads

$$\epsilon(t) = \Delta\omega\,\epsilon_{\text{dis}}(t)\,\tilde{f}(\Delta\omega\,t). \qquad (8)$$

Eq. (8) is our main result since inserting it into (2) gives the electric field pulse for an arbitrary pulse shape and general spectrum. It is a combination of a pure analytic result for the interpolation and a numeric discrete Fourier transform. The choices for the number of points M in the frequency domain and their spacing $\Delta\omega$ cannot be determined a priory on general grounds but require knowledge about $\tilde{\epsilon}$. We give examples in the following section.

APPLICATIONS

Shielded Spectra

In this section we consider a Gaussian bunch of width σ and a low-frequency cutoff due to shielding. The spectrum will be the low-frequency synchrotron radiation spectrum, $E_0(\omega) \sim \omega^{1/6}$, multiplied by a cutoff function $G(\omega)$. Based on [7] we introduce a low-frequency cutoff as

$$G(\omega) \equiv \left(1 - e^{-\omega/\omega_{\text{cut}}}\right)^n. \qquad (9)$$

The frequency ω_{cut} sets the frequency below which frequencies are exponentially suppressed and the order n determines the steepness of the transition. For either $\omega_{\text{cut}} = 0$ or $n = 0$ the spectrum is unshielded. In practice, it is more convenient to use the frequency $\omega_{50} \equiv -\ln\left(1 - 2^{-1/n}\right)\omega_{\text{cut}}$, at which the cutoff function $G(\omega)$ reaches 50%.

Fig. 1 shows the effect of the cutoff frequency ω_{50} on the electric field. The peak field decreases exponentially with increasing cutoff. For $\omega_{50} > 1/2\sigma$ an oscillating component develops as well. It can be understood by noting that for $\omega_{50} > 1/2\sigma$ the spectrum $\tilde{\epsilon}(\omega)$ can be roughly approximated by

$$\tilde{\epsilon}(\omega) \sim e^{-(\omega-\Omega)^2\Sigma^2/2}, \qquad (10)$$

which is a Gaussian with width determined by Σ and centered at frequency Ω, with $\Omega/\omega_{50} \sim 1$ and $\Sigma/\sigma \sim 1$. The Fourier transform of (10) then reads

$$\epsilon(t) \sim e^{-t^2/2\Sigma^2}\cos(\Omega t), \qquad (11)$$

and shows the oscillating term. If $\Omega\Sigma > 2\pi$ one period of the cos fits into the Gaussian envelope. We have $\Omega\Sigma \simeq 2.0$ and $\Omega\Sigma \simeq 2.6$ for $\omega_{50} = 1/2\sigma$ and $\omega_{50} = 2/\sigma$, respectively, in Fig. 1.

Fig. 2 shows the influence of the cutoff order n. Increasing n above 1 has only a minor influence on the peak field, as was also noticed in [7].
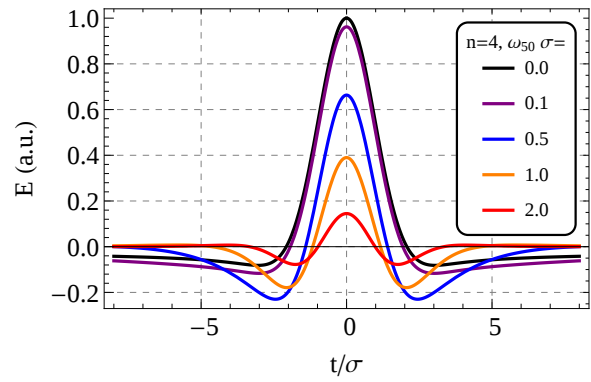
05 Beam Dynamics and Electromagnetic Fields

D06 Code Developments and Simulation Techniques

Figure 1: Normalized electric field of a Gaussian bunch over time (in units of bunch length σ) for different cutoff frequencies ω_{50} (in units of inverse bunch length). The peak field decreases exponentially with increasing ω_{50}. For the oscillation see text.
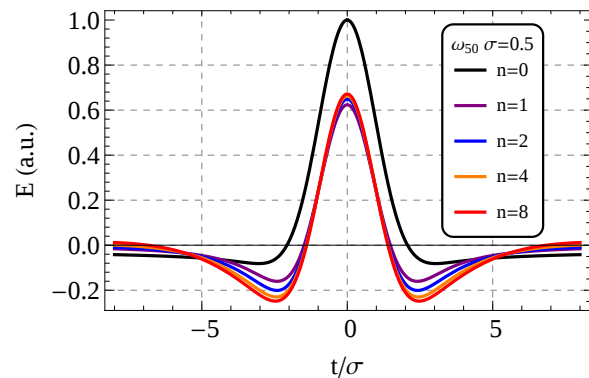

Figure 2: Normalized electric field of a Gaussian bunch over time (in units of bunch length σ) for different orders n. The unshielded case corresponds to $n = 0$. Orders $n > 1$ only have a minor influence on the peak field.

Ultra-short Bunches

We now consider bunches emitting synchrotron radiation, whose single particle spectrum is given by [3]

$$E_0(\omega) = \frac{e}{\sqrt{4\pi}\epsilon_0 c}\frac{3^{1/4}\sqrt{\gamma}}{\sqrt{2}}\sqrt{S\left(\frac{\omega}{\omega_c}\right)}, \qquad (12)$$

$$S(x) = x\int_x^\infty K_{5/3}(\xi)\,d\xi,$$

where γ and ω_c denote the Lorentz factor and critical frequency, respectively. For $\omega \ll \omega_c$, $E_0(\omega) \sim \omega^{1/6}$, as was used in the previous section. This approximation is justified as long as $1 \ll \sigma\omega_c$ [5] but breaks down for ultra-short bunches considered in this section.

When using $\tilde{\epsilon}(\omega) \sim \tilde{\rho}(\omega)\sqrt{S(\omega/\omega_c)}$ in (7) one has to properly choose $\Delta\omega$ and M. For small ω $\tilde{\epsilon}(\omega)$ increases as $\omega^{1/6}$ and for large ω is suppressed either polynomially by $\rho(\omega)$ (if $1 \ll \sigma\omega_c$) or exponentially by $\sqrt{S}$ (if $1 \gg \sigma\omega_c$). But $\omega^{1/6}$ has infinite derivatives at $\omega = 0$ and, thus, cannot be approximated by a cubic spline. However, the
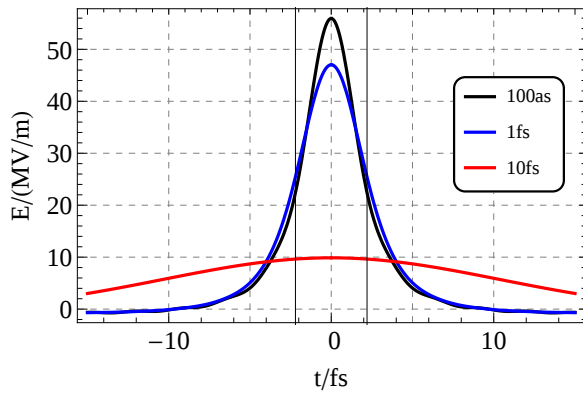
Figure 3: Field pulses for a long (10 fs) and two ultra-short Gaussian bunches (1 fs and 0.1 fs). The FWHM of the ultra-short pulses roughly equals $1/\omega_c$ (vertical lines). Decreasing the bunch length further does not change the pulse.
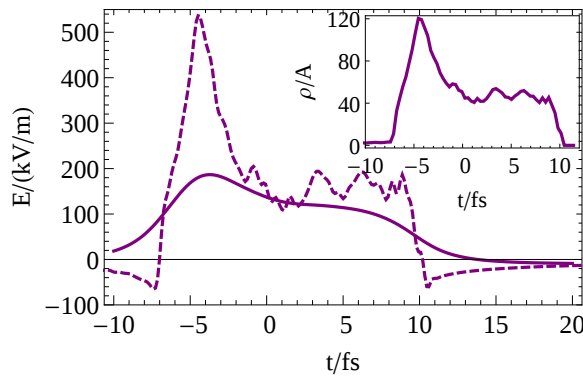

Figure 4: Calculated electric field pulse for a simulated 1 pC bunch profile (inset) at FLUTE, using the full synchrotron spectrum (solid). The use of the low-frequency spectrum (dashed) is invalid in this case. Focusing can increase the peak field by two orders of magnitude.

error becomes small for small $\Delta\omega$ and we found that

$$\Delta\omega = \begin{cases} 0.1/\sigma, & 1 \ll \sigma\omega_c \\ 0.05\omega_c, & 1 \gg \sigma\omega_c \end{cases}, \qquad (13)$$

leads to errors of a few percent.

Fig. 3 shows the pulse shapes for Gaussian bunches with $\sigma = 10$ fs, 1 fs, and 0.1 fs, corresponding to $\sigma\omega_c = 2.3$, 0.2, and 0.02, respectively. Thus, the latter two bunches are ultra-short. Decreasing the bunch length further does not change the pulse. This is because $\tilde{\rho}(\omega) \sim 1$ up to ω_c and $\tilde{\epsilon}$ becomes independent of $\tilde{\rho}$. For the same reasoning the FWHM of the ultra-short pulses depends only on ω_c and is about $1/\omega_c$ (vertical lines in Fig. 3).

For long Gaussian bunches the peak field increases as $\sigma^{-7/6}$ for decreasing σ. When the bunch becomes ultra-short the peak field approaches a constant that is only a function of the bunch charge.

Finally, we calculate the pulse of a bunch simulated for FLUTE [1, 2]. The longitudinal bunch profile of the 1 pC bunch, shown in the inset of Fig. 4, has an RMS of 5.6 fs leading to a broad $\tilde{\rho}$ extending up to $2\omega_c$. Ignoring that the high-frequency cutoff is given by ω_c is, thus, no longer justified. Doing so leads to a peak field too large by a factor of ~ 2.7, because the spectrum erroneously has more intensity at higher frequencies. Furthermore, the field pulse follows the shape of the charge profile. At large times (corresponding to small frequencies), both methods yield the same result, since their $\tilde{\epsilon}$ hardly differ. The full field pulse is "washed-out" and decreasing the bunch length further would not yield a higher peak field. Using a different radiation generating process with a broader spectrum, e.g. transition radiation, would change that.

SUMMARY

We introduced a method to calculate the electric field of coherent THz pulse for arbitrary bunch shapes and general spectra. It is a combination of a discrete Fourier transform and analytic results. As applications we discussed shielded spectra and fields of ultra-short bunches emitting synchrotron radiation. In the later case, the spectra are limited by the critical frequency, leading to "washed-out" pulses. Using emission processes that have a broader spectrum, e.g. transition radiation, would then lead to pulses whose width again is limited by the bunch length. This will be pursued in the future.

REFERENCES

[1] M.J. Nasse et al., Rev. of Sci. Instrum. **84**, 022705 (2013).

[2] M. Schuh et al., MOPRO066, Proc. IPAC2014, Dresden, Germany.

[3] H. Wiedemann, Particle Accelerator Physics, ISBN 978-3-540-49043-2, Springer (2007).

[4] A.S. Müller, Rev. Accel. Sci. Tech. **3**, 165 (2010).

[5] M. Schwarz et al., Phys. Rev. ST Accel. Beams **17**, 050701 (2014).

[6] M. Günther, B. S. thesis, Karlsruhe Institute of Technology (2014).

[7] A.-S. Müller et al., WEPC047, Proc. EPAC2008, Genoa, Italy (2008).