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BETATRON OSCILLATIONS IN PLANAR DIPOLE FIELD

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Abstract

In this paper, in preparation to the European XFEL commissioning, we consider the procedure of calculation of focusing properties of chicane-type bunch compressors and planar undulators using 2D magnetic field model (approximation of infinitely wide poles).

INTRODUCTION

In the study of the beam dynamics in undulator and wiggler magnetic fields it is a long-established practice to describe motion of particles not in the coordinate system attached to the wiggling trajectory of the beam centroid but in a (straight) Cartesian coordinate system in which equations of motion are essentially simpler and analytical or numerical analysis of their properties is more straightforward. In this paper we suggest to extend this practice on the study of the beam dynamics in an arbitrary planar dipole field including particle motion in the chicane-type bunch compressors with the benefit of the possibility of an accurate and, in the same time, relatively simple treatment of the dipole fringe effects. It is clear that this extension requires to go beyond the usual approximations well accepted for the description of the motion inside an undulator including usage of only the first and the second field integrals for the description of the beam centroid oscillations, averaging over a (short) undulator period in order to account for the natural undulator focusing and etc. So, as a first step in this direction, we provide exact (in the form of series) formulas for the motion of the beam centroid and for the nonlinear transverse and longitudinal dispersions, and derive equations of linear betatron oscillations.

EQUATIONS OF MOTION

We describe the particle motion in a Cartesian coordinate system with x, y and z as the horizontal, vertical and longitudinal direction, respectively. We assume that the longitudinal coordinate z can be introduced as an independent variable, take a pseudoparticle flying along the z-axis in the field free space as the reference particle, and use a complete set of symplectic variables $\boldsymbol{u} = (x, p_x, y, p_y, \sigma, \varepsilon)^\top$ as particle coordinates. In this set p_x and p_y are transverse canonical monenta scaled with the constant kinetic momentum of the reference particle p_0 and the variables σ and ε which describe the longitudinal dynamics are

$$\sigma = c\,\beta_0\,(t_0 - t), \qquad \varepsilon = (\mathcal{E} - \mathcal{E}_0)\,/\,(\beta_0^2\,\mathcal{E}_0), \qquad (1)$$

where $\mathcal{E}_0$, β_0 and $t_0 = t_0(z)$ are the energy of the reference particle, its velocity in terms of the speed of light c and its arrival time at a certain position z, respectively. In these variables, the Hamiltonian describing the motion of a particle in a static magnetic field takes on the form

$$H(x, p_x, y, p_y, \sigma, \varepsilon)$$
$$= \varepsilon - \sqrt{æ^2(\varepsilon) - (p_x - \bar{A}_x)^2 - (p_y - \bar{A}_y)^2} - \bar{A}_z, \qquad (2)$$

where $\bar{A} = (e/p_0)\,A$ is the normalized vector potential and

$$æ(\varepsilon) = \sqrt{(1+\varepsilon)^2 - (\varepsilon/\gamma_0)^2} = \sqrt{1 + 2\varepsilon + \beta_0^2\,\varepsilon^2}. \qquad (3)$$

In order to specify the vector potential A we have to specify first the magnetic field $B = (B_x, B_y, B_z)^\top$, which in a current-free region can be described in terms of a scalar potential Ψ (with $B = \nabla\Psi$) that satisfies the Laplace equation. Because we assume that our magnetic system is built from optical elements such that they are symmetric about the horizontal midplane $y = 0$ and that the field is homogeneous along the x-axis, it follows that the potential Ψ is an odd function of y

$$\Psi(x, y, z) \;=\; -\Psi(x, -y, z) \qquad (4)$$

and is independent from x. These properties of the scalar potential are already sufficient for the specification of the vector potential, which we will take in the following form

$$A_x \;=\; 0, \quad A_y \;=\; x\,\frac{\partial\Psi}{\partial z}, \quad A_z \;=\; -x\,\frac{\partial\Psi}{\partial y}. \qquad (5)$$

Note that expressing the potential Ψ as a formal power series in y one obtains

$$\Psi \;=\; \sum_{m=0}^{\infty} (-1)^m\, b_0^{[2m]}(z)\,\frac{y^{2m+1}}{(2m+1)!}$$
$$= b_0(z)\,y \;-\; b_0^{[2]}(z)\,\frac{y^3}{6} \;+\; b_0^{[4]}(z)\,\frac{y^5}{120} \;+\; \ldots, \qquad (6)$$

where $b_0(z)$ is distribution of the vertical magnetic field in the horizontal midplane and the index $[n]$ indicates the n-th derivative with respect to the longitudinal variable z.

MOTION IN THE SYMMETRY PLANE

Due to the symmetry relation (4) the magnetic field is perpendicular to the horizontal midplane $y = 0$, which means that the particle remains in this plane if at the beginning its vertical position and kinematic momentum were equal to zero, that is

$$y \;=\; p_y - \bar{A}_y \;=\; 0 \qquad (7)$$

for all z within the system length if these quantities were equal to zero at the system entrance position $z = 0$. The three remaining equations that define the motion of a particle on the manifold (7) become[1]

$$\frac{dx}{dz} \;=\; \frac{[p_x / æ(\varepsilon)]}{\sqrt{1 - [p_x / æ(\varepsilon)]^2}}, \qquad (8)$$

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[1] For our vector potential Eqs. (7) are equivalent to Eqs. $y = p_y = 0$.

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$$\frac{dp_x}{dz} = -h_0(z), \quad h_0(z) \stackrel{\text{def}}{=} \frac{e}{p_0} b_0(z), \tag{9}$$

$$\frac{d\sigma}{dz} = 1 - \frac{d\text{æ}(\varepsilon)}{d\varepsilon} \frac{1}{\sqrt{1-[p_x/\text{æ}(\varepsilon)]^2}}. \tag{10}$$

The solution of Eq. (9) is trivial

$$p_x(z) = p_x(0) - a(z), \quad a(z) = \int_0^z h_0(\tau)\, d\tau, \tag{11}$$

and, after its substitution into Eqs. (8) and (10), one comes to the problem of evaluation of two definite integrals, which, as we will show below, can be done in the form of series involving field integrals and Gegenbauer polynomials.

Gegenbauer Polynomials in Lee-Whiting's Notations

Gegenbauer (or ultraspherical) polynomials $C_n^{\nu}(\psi)$ can be defined in terms of their generating function

$$\left(1 - 2t\psi + t^2\right)^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(\psi)\, t^n. \tag{12}$$

For us it is convenient to introduce new polynomials following Lee-Whiting's paper on dipole fringe effect [1] via

$$D_n^m(\psi) = i^n C_n^{\frac{m}{2}}(i\psi), \tag{13}$$

where i is the imaginary unit. Then it follows from the known properties of the Gegenbauer polynomials that

$$D_0^m(\psi) = 1, \quad D_1^m(\psi) = -m\,\psi, \tag{14}$$

and that for $n \geq 2$ there is the recurrence relation

$$D_{n+2}^m(\psi) = -\tfrac{2n+2+m}{n+2}\,\psi D_{n+1}^m(\psi) + \tfrac{n+m}{n+2}\, D_n^m(\psi). \tag{15}$$

Eqs. (14) and (15) show that all $D_n^m(\psi)$ are real for the real values of their argument ψ, despite the appearance of the imaginary unit i in the defining relation (13). As other important for us properties let us mention the equalities

$$D_{2k-1}^m(0) = 0, \quad D_{2k}^m(0) = \frac{\Gamma(k+\frac{m}{2})}{\Gamma(k+1)\,\Gamma(\frac{m}{2})}, \tag{16}$$

$$D_{2k}^1(0) = -(2k-1)D_{2k}^{-1}(0) = \frac{(2k-1)!!}{2^k k!}, \tag{17}$$

where Γ is the Gamma function and $k = 1,2,3,\ldots$.

The usefulness of these polynomials for our considerations is connected with the following two important identities. Let us assume that

$$\sin(\psi_2) = \sin(\psi_1) - b. \tag{18}$$

Then

$$\sec^m(\psi_2) = \sec^m(\psi_1) \sum_{n=0}^{\infty} D_n^m\,[\tan(\psi_1)]\,\sec^n(\psi_1)\, b^n \tag{19}$$

and for $m \neq 2$ one has

$$\sin(\psi_2)\,\sec^m(\psi_2) = -\frac{\sec^{m-1}(\psi_1)}{m-2} \cdot \sum_{n=1}^{\infty} n D_n^{m-2}\,[\tan(\psi_1)]\,\sec^{n-1}(\psi_1)\, b^{n-1}. \tag{20}$$

Exact Solution in the Form of Series

Let φ_z^{ε} be the angle which the trajectory with the initial conditions $p_x(0)$ and ε makes with the z-axis. Then one can write that

$$p_x(z)/\text{æ}(\varepsilon) = \sin\left(\varphi_z^{\varepsilon}\right) = \sin\left(\varphi_0^{\varepsilon}\right) - a(z)/\text{æ}(\varepsilon). \tag{21}$$

Next, substituting Eqs. (21) into Eq. (8) and then using Eq. (20), one obtains

$$\frac{dx}{dz} = \sin\left(\varphi_z^{\varepsilon}\right)\sec\left(\varphi_z^{\varepsilon}\right)$$

$$= \sum_{n=0}^{\infty}(n+1)D_{n+1}^{-1}\left[\tan\left(\varphi_0^{\varepsilon}\right)\right]\left[\frac{\sec\left(\varphi_0^{\varepsilon}\right)}{\text{æ}(\varepsilon)}\right]^n a^n(z), \tag{22}$$

which can be easily integrated with the result that

$$x(z) = x(0) + \sum_{n=0}^{\infty}(n+1)D_{n+1}^{-1}\left[\tan\left(\varphi_0^{\varepsilon}\right)\right]\left[\frac{\sec\left(\varphi_0^{\varepsilon}\right)}{\text{æ}(\varepsilon)}\right]^n A_n(z), \tag{23}$$

where the (scaled) field integrals A_n are defined as follows

$$A_n(z) = \int_0^z a^n(\tau)\, d\tau. \tag{24}$$

By analogy

$$\sigma(z) = \sigma(0) + z - \frac{d\text{æ}(\varepsilon)}{d\varepsilon}\sum_{n=0}^{\infty} D_n^1\left[\tan\left(\varphi_0^{\varepsilon}\right)\right]\left[\frac{\sec\left(\varphi_0^{\varepsilon}\right)}{\text{æ}(\varepsilon)}\right]^n A_n(z). \tag{25}$$

MOTION OF THE BEAM CENTROID AND BETATRON OSCILLATIONS

By definition the beam centroid is a particle which has all coordinates equal to zero at the system entrance. We will denote the coordinates of the beam centroid by $\mathring{\boldsymbol{u}}$ and, according to the results of the previous section, the dynamics of the beam centroid is given by the following equations

$$\mathring{x}(z) = -\sum_{k=1}^{\infty} \tfrac{2k}{2k-1}\, D_{2k}^1(0)\, A_{2k-1}(z)$$

$$= -A_1(z) - \tfrac{1}{2}\, A_3(z) - \tfrac{3}{8}\, A_5(z) - \ldots, \tag{26}$$

$$\mathring{p}_x(z) = -a(z) \stackrel{\text{def}}{=} \sin(\mathring{\varphi}_z), \tag{27}$$

$$\mathring{\sigma}(z) = -\sum_{k=1}^{\infty} D_{2k}^1(0)\, A_{2k}(z)$$

$$= -\tfrac{1}{2}\, A_2(z) - \tfrac{3}{8}\, A_4(z) - \tfrac{5}{16}\, A_6(z) - \ldots, \tag{28}$$

$$\mathring{y}(z) = \mathring{p}_y(z) = \mathring{\varepsilon}(z) = 0. \tag{29}$$

Let us introduce new variables $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \mathring{\boldsymbol{u}}$ which describe the deviations of the solution for an arbitrary particle from the beam centroid coordinates and then linearize the equations obtained. The resulting equations (the equations of linear betatron oscillations) are governed by the quadratic Hamiltonian

$$\tilde{H}_2 = \frac{\sec^3(\mathring{\varphi}_z)}{2}\Big\{\tilde{p}_x^2 - 2\sin(\mathring{\varphi}_z)\,\tilde{p}_x\,\tilde{\varepsilon}$$

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$$+\left[1-\beta_0^2\cos^2(\overset{\circ}{\varphi}_z)\right]\tilde{\varepsilon}^2\Big\}$$

$$+\frac{1}{2}\Big\{\sec(\overset{\circ}{\varphi}_z)\left(\tilde{p}_y-h_0^{[1]}\overset{\circ}{x}\,\tilde{y}\right)^2-h_0^{[2]}\overset{\circ}{x}\,\tilde{y}^2\Big\} \quad (30)$$

and their fundamental matrix solution has the form

$$M=\begin{pmatrix} 1 & r_{12} & 0 & 0 & 0 & r_{16}\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & r_{33} & r_{34} & 0 & 0\\ 0 & 0 & r_{43} & r_{44} & 0 & 0\\ 0 & r_{52} & 0 & 0 & 1 & r_{56}\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (31)$$

where r_{km} elements related to the horizontal and longitudinal motion are given by the following expressions

$$r_{12}(z)=z+\sum_{k=1}^{\infty}(2k+1)D_{2k}^1(0)\,A_{2k}(z), \quad (32)$$

$$r_{16}(z)=r_{52}(z)=\sum_{k=1}^{\infty}2k\,D_{2k}^1(0)\,A_{2k-1}(z), \quad (33)$$

$$r_{56}(z)=\frac{z}{\gamma_0^2}+\sum_{k=1}^{\infty}\left(2k+\frac{1}{\gamma_0^2}\right)D_{2k}^1(0)\,A_{2k}(z), \quad (34)$$

and the elements related to the vertical motion have to be obtained by numerical integration of the equations of vertical betatron oscillations

$$\frac{d\tilde{y}}{dz}=\sec(\overset{\circ}{\varphi}_z)\left(\tilde{p}_y-h_0^{[1]}\overset{\circ}{x}\,\tilde{y}\right), \quad (35)$$

$$\frac{d\tilde{p}_y}{dz}=h_0^{[2]}\overset{\circ}{x}\,\tilde{y}+\sec(\overset{\circ}{\varphi}_z)\,h_0^{[1]}\overset{\circ}{x}\left(\tilde{p}_y-h_0^{[1]}\overset{\circ}{x}\,\tilde{y}\right). \quad (36)$$

To make this integration more efficiently, it is useful to introduce new variables

$$\tilde{\tilde{y}}=\tilde{y},\quad \tilde{\tilde{p}}_y=\tilde{p}_y-h_0^{[1]}\overset{\circ}{x}\,\tilde{y}, \quad (37)$$

in which Eqs. (35) and (36) are simplified to the form

$$\frac{d\tilde{\tilde{y}}}{dz}=\sec(\overset{\circ}{\varphi}_z)\,\tilde{\tilde{p}}_y, \quad (38)$$

$$\frac{d\tilde{\tilde{p}}_y}{dz}=-h_0^{[1]}\sin(\overset{\circ}{\varphi}_z)\,\sec(\overset{\circ}{\varphi}_z)\,\tilde{\tilde{y}}. \quad (39)$$

Note that Eqs. (35) and (36) are equivalent to the single second order differential equation

$$\frac{d^2\tilde{y}}{dz^2}+\left[\sin(\overset{\circ}{\varphi}_z)\sec^2(\overset{\circ}{\varphi}_z)\right]\frac{d}{dz}(h_0\,\tilde{y})=0, \quad (40)$$

which, as one can expect, in the small angle approximation turns into the equation

$$\frac{d^2\tilde{y}}{dz^2}-\left[\int_0^z h_0(\tau)\,d\tau\right]\frac{d}{dz}(h_0\,\tilde{y})=0, \quad (41)$$

which is usually used (typically, after averaging of its coefficients over an undulator period) for the estimation of the natural vertical focusing of planar undulators.

NONLINEAR DISPERSIONS

By definition the nonlinear longitudinal dispersion η_σ is the difference

$$\eta_\sigma(z,\varepsilon)=\sigma(z,\varepsilon)-\overset{\circ}{\sigma}(z)$$

$$\overset{\text{def}}{=} r_{56}(z)\,\varepsilon+r_{566}(z)\,\varepsilon^2+r_{5666}(z)\,\varepsilon^3+\ldots, \quad (42)$$

where $\sigma(z,\varepsilon)$ is the component of the solution for which all initial conditions except for a given ε are equal to zero at the system entrance. Using Eqs. (25) and (28) one can show that it is given by the following analytical expression

$$\eta_\sigma(z,\varepsilon)=\sum_{k=0}^{\infty}D_{2k}^1(0)\left[1-\frac{1}{\text{æ}^{2k}(\varepsilon)}\frac{d\text{æ}(\varepsilon)}{d\varepsilon}\right]A_{2k}(z). \quad (43)$$

The coefficients $r_{56\ldots6}$ can be obtained from Eq. (43) by expanding the expression in the square brackets with respect to the variable ε. For example

$$r_{56}(z)=A_2(z)+\tfrac{3}{2}A_4(z)+\tfrac{15}{8}A_6(z)+\ldots$$

$$+\tfrac{1}{\gamma_0^2}\left[z+\tfrac{1}{2}A_2(z)+\tfrac{3}{8}A_4(z)+\ldots\right], \quad (44)$$

$$r_{566}(z)=-\tfrac{3}{2}A_2(z)-\tfrac{15}{4}A_4(z)-\tfrac{105}{16}A_6(z)-\ldots$$

$$-\tfrac{1}{\gamma_0^2}\left[\tfrac{3}{2}z+\tfrac{9}{4}A_2(z)+\tfrac{45}{16}A_4(z)+\ldots\right], \quad (45)$$

$$r_{5666}(z)=2A_2(z)+\tfrac{15}{2}A_4(z)+\tfrac{35}{2}A_6(z)+\ldots$$

$$+\tfrac{2}{\gamma_0^2}\left[z+3A_2(z)+\ldots\right]+\tfrac{1}{2\gamma_0^4}\left[z+\tfrac{3}{2}A_2(z)+\ldots\right], \quad (46)$$

and one can see that in ultrarelativistic limit and for relatively small deflection angles the relations

$$r_{566}(z)\approx-\tfrac{3}{2}\,r_{56}(z),\quad r_{5666}(z)\approx 2\,r_{56}(z), \quad (47)$$

which are known for the hard edged model of the chicane-type bunch compressors, are valid for the particle motion in an arbitrary planar dipole field.

By analogy, for the nonlinear transverse dispersion one also obtains the following analytical formula

$$\eta_x(z,\varepsilon)=x(z,\varepsilon)-\overset{\circ}{x}(z)$$

$$\overset{\text{def}}{=} r_{16}(z)\,\varepsilon+r_{166}(z)\,\varepsilon^2+r_{1666}(z)\,\varepsilon^3+\ldots$$

$$=\sum_{k=1}^{\infty}\tfrac{2k}{2k-1}D_{2k}^1(0)\left[1-\frac{1}{\text{æ}^{2k-1}(\varepsilon)}\right]A_{2k-1}(z). \quad (48)$$

REFERENCES

[1] G.E.Lee-Whiting, "First- and second-order motion through the fringing field of a bending magnet", Nucl. Instr. and Meth. A294 (1990) 31-71.