

ANALYTICAL APPROACH TO THE BEAM-BEAM INTERACTION WITH THE HOUGLASS EFFECT

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Abstract

The hourglass effect arises due to a coupling between the longitudinal and transverse bunch planes. This coupling will result in a charge density distribution that will vary parabolically through the Interaction Point (IP). Here a method of analytically determining the electric field a particle receives from a charge density distribution which varies parabolically when centred at the IP, is derived for a 2D transverse model of a Gaussian bunch.

INTRODUCTION.

The High Luminosity Large Hadron Collider (HL-LHC) seeks to reach even higher luminosities than previously achieved. An increased luminosity can lead to a saturation or "pile up" in the machine detectors. To avoid pile up in the detectors, luminosity levelling has been proposed which aims to hold the luminosity constant over the duration of a physics run. One proposed method of levelling the luminosity is to reduce the β -function at the IP as the bunch intensity decays due to proton burn off.

One of the benefits of levelling by this method is it is operationally easy to implement and the longitudinal vertex density remains fixed when there is no crossing angle. The flat beam option is an alternative operational scenario for the HL-LHC when there are no crab cavities installed. To reach high luminosities without crab cavities, small β^* is required to generate high luminosities. The final levelling step of the flat beam option will give a β -function at the IP of $\beta_{x,y}^* = 0.3/0.075$ m. In this scenario, the β -function in the vertical plane at the end of a physics run will be comparable to the length of the bunch. When the length of the bunch is comparable to the transverse bunch size, a coupling between the planes is introduced, which results in the bunch varying parabolically at the IP. This coupling will result in a deviation from a Gaussian distribution in the longitudinal coordinate as the bunch passes through the IP. Such a deviation will result in some particles not colliding at the minimum β^* , which will hence lead to a reduction in luminosity. This is known as the hourglass effect.

Here, a new approach is applied to the beam-beam interaction to obtain analytical solutions for the electric field of a bunch that is centred at the IP undergoing a parabolic variation due to the hourglass effect. From the electric field in the rest frame of the bunch, one can later obtain the force a test particle experiences when the counter rotating bunch is centred at the IP.

THEORY

Starting from Maxwell's equations, which when expressed in full are given by,

$$\partial_y B_z - \partial_z B_y = \mu (J_x + \epsilon \partial_t E_x), \quad (1)$$

$$\partial_x B_z - \partial_z B_x = -\mu (J_y + \epsilon \partial_t E_y), \quad (2)$$

$$\partial_x B_y - \partial_y B_x = \mu (J_z + \epsilon \partial_t E_z), \quad (3)$$

$$\partial_y E_z - \partial_z E_y = -\partial_t B_x, \quad (4)$$

$$\partial_x E_z - \partial_z E_x = \partial_t B_y, \quad (5)$$

$$\partial_x E_y - \partial_y E_x = -\partial_t B_z, \quad (6)$$

$$\partial_x B_x + \partial_y B_y + \partial_z B_z = 0, \quad (7)$$

$$\partial_x E_x + \partial_y E_y + \partial_z E_z = \frac{\rho(x,y,z)}{\epsilon_0}. \quad (8)$$

If these equations are written in the rest frame of the bunch, the problem can be treated as electrostatic, assuming that there are no external magnetic fields in the rest frame. This provides an appropriate starting point and is one that is applied in the original derivation given by Bassetti-Erskine [1]. Maxwell's equations in the transverse planes are then given by

$$\partial_x E_y - \partial_y E_x = 0, \quad (9)$$

$$\partial_x E_x + \partial_y E_y = \frac{\rho(x,y)}{\epsilon_0}, \quad (10)$$

Since the charge density distribution, ρ , is known and is given by a Gaussian distribution, one is able to solve this system of equations to obtain the transverse electric field experienced by a particle as it traverses the electric field of a counter rotating bunch. Following the method proposed by Muratori *et al*, [2], one attempts to solve these non-linear coupled partial differential equations using an expression with a number of unconstrained functions, which in turn, will reduce to the well known Bassetti-Erskine (BE) equation. This method can then be extended for a charge density distribution that has a parabolic dependence on the longitudinal coordinate.

The BE equation for radially symmetric beams with fixed transverse beam sizes is well known and is obtained through solving Poisson's equation for a Gaussian source. Poisson's equation relating the scalar potential φ to the charge density distribution is given by,

$$\nabla^2 \varphi = \frac{\rho}{\epsilon_0}. \quad (11)$$

Solving this using the Green's function method as introduced by S.Kheifets [3] allows the radial electric field to be

obtained,

$$E_r = \frac{nq}{2\pi\epsilon_0 r} \left[1 - e^{-\frac{r^2}{2\sigma_r^2}} \right]. \quad (12)$$

This gives the definite form of the BE equation for radially symmetric round beams. Following the approach in [2], we now use the method of unconstrained functions to derive equation 12. An appropriate expression is chosen which with unconstrained functions can reduce to the well known BE equation. An expression for the electric field is selected in the form,

$$E_x = f_1(x, y) \left[1 - e^{f_2(x, y)} \right], \quad (13)$$

$$E_y = f_3(x, y) \left[1 - e^{f_2(x, y)} \right], \quad (14)$$

where x and y are the transverse coordinates. From these equations one can seek to solve for the unknown functions $f_{1,3}$, when the bunch distribution is known. Differentiating equations 13, 14 with respect to all variables gives,

$$\partial_x E_y = \partial_x f_3 \left[1 - e^{f_2} \right] - f_3 \partial_x f_2 e^{f_2}, \quad (15)$$

$$\partial_x E_x = \partial_x f_1 \left[1 - e^{f_2} \right] - f_1 \partial_x f_2 e^{f_2}, \quad (16)$$

$$\partial_y E_x = \partial_y f_1 \left[1 - e^{f_2} \right] - f_1 \partial_y f_2 e^{f_2}, \quad (17)$$

$$\partial_y E_y = \partial_y f_3 \left[1 - e^{f_2} \right] - f_3 \partial_y f_2 e^{f_2}. \quad (18)$$

These expressions can then be substituted into equations 9 and 10 to give,

$$\left[1 - e^{f_2} \right] (\partial_x f_y - \partial_y f_1) + e^{f_2} \left[f_1 \partial_y f_2 - f_3 \partial_x f_2 \right] = 0, \quad (19)$$

$$\left[1 - e^{f_2} \right] (\partial_x f_1 + \partial_y f_3) + e^{f_2} \left[f_3 \partial_y f_2 - f_1 \partial_x f_2 \right] = \rho, \quad (20)$$

Collecting powers of the exponential argument,

e^0

$$\partial_x f_3 - \partial_y f_1 = 0, \quad (21)$$

$$\partial_x f_1 + \partial_y f_3 = 0, \quad (22)$$

e^{f_2} .

$$\left[-e^{f_2} \right] \left[\partial_x f_3 - \partial_y f_1 \right] + e^{f_2} \left[f_1 \partial_y f_2 - f_3 \partial_x f_2 \right] = 0, \quad (23)$$

$$\left[-e^{f_2} \right] \left[\partial_x f_1 + \partial_y f_3 \right] - e^{f_2} \left[f_3 \partial_y f_2 + f_1 \partial_x f_2 \right] = \rho. \quad (24)$$

From these equations, one is able to simplify and hence solve for the unconstrained functions $f_{1,3}$. Due to the brevity of this paper only the simplest case is considered here such that the bunch is centred at the IP and the collision is head on. The charge density distribution is known, and is given by a Gaussian function, in this case defined as $\rho = \rho_0 \cdot e^{f_2}$,

where ρ_0 is the normalised charge density distribution and equates to $\rho_0 = \frac{nq}{2\pi\sigma_r^2}$.

Making a substituting for f_3 in terms of f_1 and substituting into equation 24 gives,

$$f_1 \left[\partial_x f_2 + \frac{(\partial_y f_2)^2}{\partial_x f_2} \right] = -1, \quad (25)$$

which can be solved to give the functions f_1 and from symmetry f_3 , for round bunches in the transverse planes.

$$f_1 = \frac{x}{r^2}, \quad f_3 = \frac{y}{r^2}, \quad (26)$$

where $r^2 = x^2 + y^2$. Substituting this back into the original form given by equation 13 for the transverse electric field gives,

$$E_x = \frac{nq}{2\pi\epsilon_0} \cdot \frac{x}{r^2} \left[1 - e^{-r^2} \right], \quad (27)$$

$$E_y = \frac{nq}{2\pi\epsilon_0} \cdot \frac{y}{r^2} \left[1 - e^{-r^2} \right]. \quad (28)$$

From equations 27 and 28, one can transform the electric field in x and y , to radial coordinates and the result is shown to reduce to equation 12.

The Hourglass Effect

When the transverse bunch size is comparable to the length of the bunch, a coupling is induced which will cause the transverse bunch size to vary parabolically at the IP. This coupling will cause a variation in the charge density distribution through the IP when $\sigma_{x,y} \sim \sigma_z$. The transverse bunch size will vary parabolically along the longitudinal coordinate. For the hourglass effect, this variation and can be given by $\sigma_i = k_i \left(1 + \frac{z^2}{\beta^{*2}} \right)$, where $k_i = \epsilon_i \cdot \beta_i^*$ and ϵ_i is the emittance of the transverse planes, β_i^* is the β -function at the IP and the transverse coordinates are given by $i = x, y$. This deviation from a Gaussian distribution along the longitudinal coordinate, will result in a test particle experiencing a different force from equation 12. When this coupling is considered the force that a test particle experiences will not only be dependent on the transverse amplitude of the test particle but also on its longitudinal distance from the IP.

Starting from equations 21 to 24 and treating as colliding head on only, one obtains the same expression as equation 25, but the function f_2 has the coupled bunch size included, such that $f_2 = -\frac{x^2}{2k_x \left(1 + \frac{z^2}{\beta^{*2}} \right)} - \frac{y^2}{2k_y \left(1 + \frac{z^2}{\beta^{*2}} \right)}$. Which for round beams will give,

$$f_1 = \frac{x \cdot k \left(1 + \frac{z^2}{\beta^{*2}} \right)}{r^2}, \quad f_3 = \frac{y \cdot k \left(1 + \frac{z^2}{\beta^{*2}} \right)}{r^2}. \quad (29)$$

Substituting $f_{1,3}$ into equation 13 and equation 14 will give the electric field in the transverse planes in the rest frame of

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the bunch as

$$E_x = \frac{nq}{2\pi\sigma_{x0}^2\epsilon_0} \cdot \frac{x \cdot k \left(1 + \frac{z^2}{\beta^{*2}}\right)}{r^2} \left[1 - \exp\left(-\frac{r^2}{2k \left(1 + \frac{z^2}{\beta^{*2}}\right)}\right) \right], \quad (30)$$

$$E_y = \frac{nq}{2\pi\sigma_{y0}^2\epsilon_0} \cdot \frac{y \cdot k \left(1 + \frac{z^2}{\beta^{*2}}\right)}{r^2} \left[1 - \exp\left(-\frac{r^2}{2k \left(1 + \frac{z^2}{\beta^{*2}}\right)}\right) \right]. \quad (31)$$

From the electric field expressions in equations 30 and 31 one can calculate the electric field in cylindrical components. Fig 1, shows how the charge density distribution and the radial electric field will vary for a test particle as it traverses a bunch with a transverse size that varies parabolically when centred at the IP.

DISCUSSION

In this paper the method of unconstrained functions as described in [2], is applied to calculate the electric field in the rest frame of the bunch experienced by a transversing test particle. The standard BE equation in radial coordinates, given by equation 12 with constant transverse beam sizes is recovered. This method is then further applied to obtain the transverse electric field experienced by a test particle when the transverse bunch size ($\sigma_{x,y}$), is dependent on the longitudinal coordinate. With a bunch size dependent on the longitudinal coordinate the electric field in the rest frame of the bunch is obtained and showed that the charge density distribution will deviate from a Gaussian along the longitudinal coordinate. This deviation will result in particles not colliding at the minimum β^* , which in turn will result in a loss of luminosity.

This method can be further applied to situations where the charge density distribution will change through the IP and may allow analytical expressions to be determined. A full derivation in three dimensions to include a crossing angle with a longitudinal electric field will be published by the authors [4].

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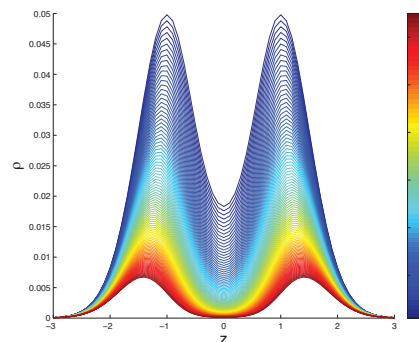
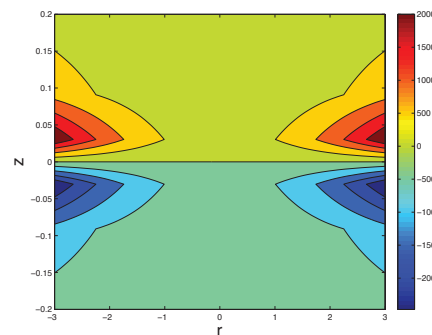
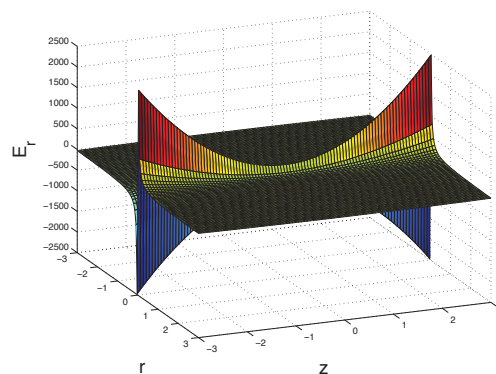


Figure 1: The top figure shows the variation of the radial electric field over the longitudinal coordinate. The middle figure shows that the magnitude of the radial E-field imparted on a test particle is dependent on both the longitudinal position and the radial amplitude of the particle. The bottom figure shows how the charge density distribution will vary with different particle amplitudes in the radial plane. The colour map determines the size of the electric field experienced by a traversing test particle. In the top and middle plots dark blue and dark red are equal in magnitude and opposite in direction. In the bottom plot blue to red indicate an increasing radial amplitude of the particle. All plots are in arbitrary units and indicate the functional form of the derived expressions.

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