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A GENERIC FORMULATION FOR EMITTANCE AND LATTICE FUNCTION EVOLUTION FOR NON-HAMILTONIAN SYSTEMS WITH STOCHASTIC EFFECTS

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Abstract

I describe a generic formulation for the evolution of emittances and lattice functions under arbitrary, possibly non-Hamiltonian, linear equations of motion. The average effect of stochastic processes, which would include ionization interactions and synchrotron radiation, is also included. I first compute the evolution of the covariance matrix, then the evolution of emittances and lattice functions from that. I examine the particular case of a cylindrically symmetric system, which is of particular interest for ionization cooling.

INTRODUCTION

I describe a general formulation for the evolution of the first and second moments of a beam distribution. Similar formulations have been presented before ([1–3] are some examples). What is of interest here is the definition of the stochastic behavior in terms of probabilities, the direct computation of the evolution of generalized lattice functions (really the symplectic normalizing transformation of the second moment matrix) and emittances, and the definition of a metric for mismatch of those lattice functions.

MATHEMATICAL FORMULATION

$\psi(z, s)$ is the distribution function for particles in the phase space coordinates z at a point s along a reference curve. We define the first and second moments of this distribution

$$\boldsymbol{a}(s) = \int z\,\psi(z, s)\,dz \tag{1}$$

$$\Sigma(s) = \int [z - \boldsymbol{a}(s)][z - \boldsymbol{a}(s)]^T \psi(z, s)\,dz \tag{2}$$

The deterministic motion of a particle is described by

$$\frac{dz}{ds} = \boldsymbol{f}(z, s) \tag{3}$$

The stochastic part of the motion is described such that $\rho(\boldsymbol{x}, z, s)d\boldsymbol{x}\,ds$ is the probability that, for a particle at z in phase space and in the interval $[s, s + ds)$, the particle is displaced in phase space by a value in the phase space volume of size $d\boldsymbol{x}$ at $\boldsymbol{x}$. Then the continuity equation is

$$\frac{\partial\psi}{\partial s} + \boldsymbol{\nabla}\cdot[\psi(z, s)\boldsymbol{f}(z, s)] = \int \psi(z - \boldsymbol{x}, s)\rho(\boldsymbol{x}, z - \boldsymbol{x}, s)\,d\boldsymbol{x} - \psi(z, s)\int \rho(\boldsymbol{x}, z, s)\,d\boldsymbol{x}. \tag{4}$$

From this one can determine the evolution of the moments. The system acts as though it were governed by a deterministic vector field $\boldsymbol{g}$ such that

$$\frac{dz}{ds} = \boldsymbol{g}(z, s) = \boldsymbol{f}(z, s) + \int \boldsymbol{x}\,\rho(\boldsymbol{x}, z, s)\,d\boldsymbol{x} \tag{5}$$

Then

$$\frac{d\boldsymbol{a}}{ds} = \int \boldsymbol{g}(z, s)\psi(z, s) \tag{6}$$

$$\frac{d\Sigma}{ds} = \int [z - \boldsymbol{a}(s)]\boldsymbol{g}(z, s)^T\psi(z, s)\,dz + \int \boldsymbol{g}(z, s)[z - \boldsymbol{a}(s)]^T\psi(z, s)\,dz + \int \boldsymbol{x}\boldsymbol{x}^T\rho(\boldsymbol{x}, z, s)\psi(z, s)\,d\boldsymbol{x}\,dz \tag{7}$$

If $\boldsymbol{g}(z, s) = \boldsymbol{g}_0(z) + JH(s)z$, with J the antisymmetric symplectic metric (H is symmetric only for a Hamiltonian system),

$$\frac{d\boldsymbol{a}}{ds} = \boldsymbol{g}_0(s) + JH(s)\boldsymbol{a}(s) \tag{8}$$

$$\frac{d\Sigma}{ds} = JH(s)\Sigma(s) - \Sigma(s)H^T(s)J + \int \boldsymbol{x}\boldsymbol{x}^T\rho(\boldsymbol{x}, z, s)\psi(z, s)\,d\boldsymbol{x}\,dz \tag{9}$$

As long as Σ is positive definite (its definition insures that it is positive *semi*-definite), then one can find a symplectic A such that

$$\Sigma(s) = A(s)E(s)A^T(s) \tag{10}$$

where E is diagonal with pairs of equal diagonal elements, which are the emittances. A contains the generalized versions of the Courant-Snyder functions that describe the *distribution*. For a distribution "matched" to a lattice, A will by definition refer to the generalization of the corresponding functions for the *lattice*. If the emittances are distinct, the right hand side of A can be multiplied by any block-diagonal rotation with 2×2 blocks (there is more freedom when some emittances are equal).

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The evolution of E and A is computed from

$$\frac{dE}{ds} = B(s) + JC(s)E(s) - E(s)C(s)J \tag{11}$$

$$B(s) = A^{-1}(s)\frac{d\Sigma}{ds}A^{-1^T}(s) \tag{12}$$

$$C(s) = A^T(s)J\frac{dA}{ds} = JA^{-1}\frac{dA}{ds} \tag{13}$$

B and C are symmetric (C because A is symplectic). $d\Sigma/ds$ in Eq. (12) is computed from Eq. (7). This equation can be split into 2×2 blocks (corresponding to the identical diagonal pairs in E), giving the solution

$$\frac{d\epsilon_i}{ds} = \frac{1}{2}\operatorname{Tr} B_{ii} \tag{14}$$

$$C_{ij} = \begin{cases} \dfrac{JB_{ij} - B_{ij}J}{4\epsilon_i} + \xi_{ii}I & i = j \\ \dfrac{JB_{ij} - B_{ij}J}{4\epsilon_i} + \xi_{ij}I + \eta_{ij}J & i \neq j \wedge \epsilon_i = \epsilon_j \\ \dfrac{\epsilon_i B_{ij}J + \epsilon_j JB_{ij}}{\epsilon_j^2 - \epsilon_i^2} & i \neq j \wedge \epsilon_i \neq \epsilon_j \end{cases} \tag{15}$$

where the ξ_{ij} are arbitrary constants. The B_{ij} and C_{ij} are 2×2 blocks. The freedom of choice for ξ_{ii} corresponds to the rotational degree of freedom in A. The freedom of choice for ξ_{ij} and η_{ij} for $i \neq j$ reflects the freedom to mix eigenvectors that have identical eigenvalues. When $\epsilon_i = \epsilon_j$ and $i \neq j$, A must (and can) be chosen to make B_{ij} traceless and symmetric.

One important application of this formalism is to matching. Say one can find a $\Sigma_L(s)$ solving Eq. (7) for a given lattice that has some desired property (a periodic solution for a ring, a desired phase space distribution at a given position, etc.). One can find a corresponding $A_L(s)$ satisfying Eq. (10), and the corresponding $E_L(s)$ will be constant (for a Hamiltonian system) or slowly and monotonically changing (for a system with damping or stochastic excitation). If one uses a particular Σ_B as an initial condition for Eq. (7), one can also find a corresponding $A_B(s)$ satisfying Eq. (10). This distribution is matched if $A_L^{-1}(s)\Sigma_B(s)A_L^{-1^T}(s)$ has slow, monotonic variation similar to $E_L(s)$. $E_B(s)$ will have a slow, monotonic variation for the same reason that $E_L(s)$ does. If $\Delta = A_L^{-1}A_B$ is a matrix with 2×2 rotation blocks, this will achieve that matching. One can thus define a metric for the degree of mismatch as

$$\sum_i \frac{\lambda_{ii}}{2}[(\Delta_{2i,2i} - \Delta_{2i+1,2i+1})^2 + (\Delta_{2i,2i+1} + \Delta_{2i+1,2i})^2] + \sum_{i \neq j} \frac{\lambda_{ij}}{2}(\Delta_{2i,2j}^2 + \Delta_{2i,2j+1}^2 + \Delta_{2i+1,2j}^2 + \Delta_{2i+1,2j+1}^2) \tag{16}$$

where $\lambda_{ij} > 0$ can be freely chosen.

Rotational Symmetry

Now consider only the transverse degrees of freedom, and assume the system is unchanged under rotations about the longitudinal axis. The covariance matrix then takes the form

$$\Sigma_R = \begin{bmatrix} \sigma_{xx} & \sigma_{xp} & 0 & L/2 \\ \sigma_{xp} & \sigma_{pp} & -L/2 & 0 \\ 0 & -L/2 & \sigma_{xx} & \sigma_{xp} \\ L/2 & 0 & \sigma_{xp} & \sigma_{pp} \end{bmatrix} \tag{17}$$

The covariance matrix is diagonalized by

$$\begin{bmatrix} \sqrt{\dfrac{\sigma_{xx}}{2\epsilon}} & 0 & 0 & \sqrt{\dfrac{\sigma_{xx}}{2\epsilon}} \\ \dfrac{\sigma_{xp}}{\sqrt{2\epsilon\sigma_{xx}}} & \sqrt{\dfrac{\epsilon}{2\sigma_{xx}}} & -\sqrt{\dfrac{\epsilon}{2\sigma_{xx}}} & \dfrac{\sigma_{xp}}{\sqrt{2\epsilon\sigma_{xx}}} \\ 0 & \sqrt{\dfrac{\sigma_{xx}}{2\epsilon}} & \sqrt{\dfrac{\sigma_{xx}}{2\epsilon}} & 0 \\ -\sqrt{\dfrac{\epsilon}{2\sigma_{xx}}} & \dfrac{\sigma_{xp}}{\sqrt{2\epsilon\sigma_{xx}}} & \dfrac{\sigma_{xp}}{\sqrt{2\epsilon\sigma_{xx}}} & \sqrt{\dfrac{\epsilon}{2\sigma_{xx}}} \end{bmatrix} \tag{18}$$

where $\epsilon^2 = \sigma_{xx}\sigma_{pp} - \sigma_{xp}^2$ and L is notably absent. The resulting emittances are $\epsilon + L/2$ and $\epsilon - L/2$. When $L = 0$, other diagonalizations are possible.

H will have the block form

$$H_R = \begin{bmatrix} H_{xx} & H_{xy} \\ -H_{xy} & H_{xx} \end{bmatrix} \tag{19}$$

For a Hamiltonian system, H_{xx} is symmetric and H_{xy} is antisymmetric.

The mismatch Δ is

$$\frac{\sigma_{Lpp}\sigma_{Bxx} + \sigma_{Bpp}\sigma_{Lxx} - 2\sigma_{Lxp}\sigma_{Bxp} - 2\epsilon_L\epsilon_B}{\epsilon_L\epsilon_B} \tag{20}$$

which is just twice (since the mismatch includes both modes) what one would obtains for the usual measure of emittance increase [4] from mismatch in a single plane. Angular momentum does not enter into the diagonalizing matrix or the mismatch.

IONIZATION COOLING

For the case of ionization cooling with uniform slabs, $\rho(\boldsymbol{x}, \boldsymbol{z}, s)$ takes on the form

$$\delta(\boldsymbol{x}_q)\frac{p(z_E)}{p_z(z_E, \boldsymbol{z}_{p_{K\perp}})}\left[\rho_{\mathrm{MS}}(\boldsymbol{x}_{p_\perp}, \boldsymbol{z}, s)\delta(x_E) + \rho_{\mathrm{dE}}(x_E, \boldsymbol{z}, s)\delta\left(\boldsymbol{x}_{p_\perp} - \frac{x_E E(z_E)}{c^2 p^2(z_E)}\boldsymbol{z}_{p_{K\perp}}\right)\right] \tag{21}$$

The two terms in brackets arise from multiple scattering (the first term) and energy loss and energy straggling (the second term). The dependency on the phase space variables $\boldsymbol{z}$ (only time will not come into play) and s takes into account only the spatial placement of material and the length of the material traversed depending on the particle trajectory. E, p, and p_z are functions that give energy, total momentum, and longitudinal momentum. The energy is a function of the energy phase space variable in case the energy phase space variable is an offset from a (possibly changing) reference energy.

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Subscripts on $\boldsymbol{x}$ and $\boldsymbol{z}$ indicate that a subset of the components of the phase space vector should be used. The $p_\perp$ subscript refers to two-dimensional transverse momentum; the additional K subscript says to use the kinetic momentum vector; the q subscript refers to the three-dimensional vector of coordinates.

Next, define

$$m_1(z,s) = -\int x_E \rho_{\mathrm{dE}}(x_E,z,s)\,dx_E \tag{22}$$

$$\approx m_{10}(s) + m_{11}(s) z_E \tag{23}$$

$$m_2(z,s) = \int x_E^2 \rho_{\mathrm{dE}}(x_E,z,s)\,dx_E \approx m_{20}(s) \tag{24}$$

z_E, the energy phase space variable, is a deviation from a central or reference energy. m_1 is the average energy loss per unit length; details of the behavior of m_1 and m_2 are given in [5]. Furthermore, for the purpose of computing matrix elements, I assume that the relationship between kinetic and canonical momenta is given by

$$p_{xK} = p_x + \frac{zeB_s(s)}{2} y \quad p_{yK} = p_y - \frac{zeB_s(s)}{2} x \tag{25}$$

with z being the charge in units of the electron charge and B_s the longitudinal field on axis. Then the contribution of the absorber to H is rotationally symmetric and given by the submatrices

$$H_{xx} = \begin{bmatrix} 0 & \frac{m_{10}}{\beta cp} \\ 0 & 0 \end{bmatrix} \qquad H_{xy} = \begin{bmatrix} \frac{m_{10} zeB_s}{2\beta cp} & 0 \\ 0 & 0 \end{bmatrix} \tag{26}$$

and there is an additional time-energy block in H of

$$\begin{bmatrix} 0 & -m_{11} \\ 0 & 0 \end{bmatrix} \tag{27}$$

Here βc is a reference velocity (typically corresponding to the average distribution energy) and p is the corresponding momentum. Finally, the last term in Eq. (9) will be, neglecting some small terms, a matrix with diagonal elements

$$0, S_{\mathrm{MS}}, 0, S_{\mathrm{MS}}, 0, m_{20} \tag{28}$$

$$S_{MS}(s) = \left(\frac{13.6\ \mathrm{MeV}}{\beta c} z\right)^2 \frac{\rho(s)}{X_0(s)} \tag{29}$$

where $X_0(s)$ is the radiation length and ρ is the material density [5].

This analysis is not always adequate for multiparticle simulations. Eq. (28) is inadequate due to significant tails in the scattering and straggling distributions. Furthermore, the terms above have neglected geometrical absorber shaping that can be used to couple longitudinal and transverse motion; this can be approximated in this analysis by including a transverse variation in ρ_{dE} and ρ_{MS}.

Straight Solenoid Channel

The primary intent of this analysis is to permit the tracking of lattice functions and emittances to aid in beamline optimization at the design stage. In this subsection, however, I will use the technique to study one particular phenomenon: the longitudinal magnetic field in ionization cooling absorbers. I will focus on a cylindrically symmetric system.

Using Eqs. (14), (18), (26), and (28), I obtain the evolution of the emittances:

$$\frac{d(\epsilon \pm L/2)}{ds} = -\frac{m_{10}}{\beta cp}\left(1 \mp \frac{zeB_s}{2}\frac{\sigma_{xx}}{\epsilon}\right)(\epsilon \pm L/2) + \frac{S_{\mathrm{MS}}}{2}\frac{\sigma_{xx}}{\epsilon} \tag{30}$$

The emittance only changes in absorbers. σ_{xx} and ϵ are not in general constant, but for the purposes of this discussion, we assume that the absorber is short enough that the σ_{xx}/ϵ remains sufficiently constant, that the magnetic field maintains σ_{xx}/ϵ approximately constant, or that an average value of that ratio can be used.

An important quantity is $r = zeB_s\sigma_{xx}/(2\epsilon)$. In a constant magnetic field with matched beam, $|r| = 1$. This leads one of the emittances to grow linearly without bound. There are two ways this can be addressed: the first is to construct a lattice so as to make $r < 1$. This requires a variation in the magnetic field, thereby reducing the energy acceptance and dynamic aperture [6], but also improves the equilibrium emittance, due to the last term in Eq. (30). Second, one periodically changes the sign of the magnetic field; this causes the mode that was previously damped more slowly to be damped more rapidly, and vice versa. If the reversal is frequent enough, the system will behave as though r were 0.

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