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RADIATION OF A PARTICLE MOVING ALONG A HELICAL TRAJECTORY IN A SEMI-INFINITE CYLINDRICAL WAVEGUIDE*

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Abstract

The radiation field of a particle which suddenly appears in an ideal waveguide and moves on a helical trajectory under the influence of external magnetic fields is calculated. The shape and character of the front of the propagating wave is determined.

INTRODUCTION

A combination of a waveguide and a helical undulator transforms the helical undulator radiation spectrum from continuous to discrete and thus improves the characteristics of its radiation significantly [1]. Usually, the stationary motion of a particle in an infinite rectangular [2-4] or circular [1, 5-9] waveguide is considered, which ignores the injection phenomenon, i.e. the instantaneous appearance of a particle at a certain point in the waveguide (some aspects of this problem are considered in [10-11]). In the present work, the problem of the stationary motion of a point particle with a charge varying with time and performing a helical motion in an infinite ideal cylindrical waveguide is considered. On this basis, the problem of a particle that suddenly appears at a certain moment of time and moves along a helical trajectory in the same waveguide is solved. In conclusion, a formula is derived that describes the gradual appearance of a bunch of charged particles, which simulates the process of its injection.

CHARGE VARYING IN TIME

Consider a relativistic point charge with longitudinal velocity V and charge $Q(t)$, with an arbitrary time dependence, moving in a homogeneous waveguide along a helical trajectory, with a revolution frequency ω_b. The waveguide is assumed to be circular with a radius b and has perfectly conducting walls. The charge density ρ and current $\vec{j}$ are given in the forms:

$$\rho(r,\varphi,z,t) = qQ(t)\,\frac{\delta(r-a)}{\sqrt{ra}}\,\delta(\varphi-\omega_b t)\delta(z-Vt)$$

$$\vec{j}(r,\varphi,z,t) = (\omega_b a\vec{e}_\varphi + V\vec{e}_z)\rho(r,\varphi,z,t) \qquad (1)$$

where $\vec{e}_\varphi$, $\vec{e}_z$ are unit vectors in cylindrical coordinates and a is the radius of the particle orbit. q is the elementary charge. The radiation field is determined from the wave equations:

$$\left\{\Delta - \frac{1}{c^2}\frac{\partial}{\partial t^2}\right\}\vec{E} = \mu_0\left\{\frac{\partial \vec{j}}{\partial t} + c^2\nabla\rho\right\}$$

$$\left\{\Delta - \frac{1}{c^2}\frac{\partial}{\partial t^2}\right\}\vec{H} = -rot\vec{j} \qquad (2)$$

with the magnetic permeability of vacuum μ_0.

In the time-frequency domain the electrical and magnetic fields are sought in the form of cylindrical mode compositions, which combine TM and TE harmonics:

$$\vec{E} = \sum_{n,m=1}^{\infty}\{\vec{E}_{nm}^{TM} + \vec{E}_{nm}^{TE}\} \qquad (3)$$

The longitudinal and transverse components of electric and magnetic harmonics are written in the form of expansion terms in Bessel functions of the first kind:

$$\tilde{E}_{nm_z}^{TM} = U_{nm}\psi_{nm}^{TM},$$

$$\psi_{nm}^{TM} = J_n\left(j_{nm}\frac{r}{b}\right)e^{jn(\varphi-\omega_0 t)}e^{jk(z-Vt)},$$

$$\tilde{H}_{nm_z}^{TE} = c\varepsilon_0 W_{nm}\psi_{nm}^{TE},$$

$$\psi_{nm}^{TE} = J_n\left(\nu_{nm}\frac{r}{b}\right)e^{jn(\varphi-\omega_0 t)}e^{jk(z-Vt)}$$

$$\vec{\mathcal{E}}_{nm}^{TM} = \{\mathcal{E}_{nm_r}^{TM}, \mathcal{E}_{nm_\varphi}^{TM}, 0\}, \qquad \vec{\mathcal{H}}_{nm}^{TE} = \{\mathcal{H}_{nm_r}^{TE}, \mathcal{H}_{nm_\varphi}^{TE}, 0\}$$

$$\vec{\tilde{E}}_{nm_t}^{TM} = A_{nm}\vec{\mathcal{E}}_{nm}^{TM}, \qquad \vec{\tilde{H}}_t^{TE} = D_{nm}\,\vec{\mathcal{H}}_{nm}^{TE}$$

$$\vec{\tilde{H}}_{nm_t}^{TM} = C_{nm}c\varepsilon_0[\vec{e}_z \times \vec{\mathcal{E}}_{nm}^{TM}]$$

$$\vec{\tilde{E}}_t^{TE} = -B_{nm}(c\varepsilon_0)^{-1}[\vec{e}_z \times \vec{\mathcal{H}}_{nm}^{TE}]$$

$$\vec{\mathcal{H}}_{nm}^{TE} = \{\mathcal{H}_{nm_r}^{TE}, \mathcal{H}_{nm_\varphi}^{TE}, 0\}$$

$$\vec{\tilde{H}}_t^{TE} = D_{nm}\,\vec{\mathcal{H}}_{nm}^{TE}$$

$$\mathcal{E}_{nm_r}^{TM} = \frac{\partial\,\psi_{nm}^{TM}}{\partial r}, \qquad \mathcal{E}_{nm_\varphi}^{TM} = j\frac{n}{r}\psi_{nm}^{TM}$$

$$\mathcal{H}_{nm_r}^{TE} = \frac{\partial\,\psi_{nm}^{TE}}{\partial r}, \qquad \mathcal{H}_{nm_\varphi}^{TE} = j\frac{n}{r}\,\psi_{nm}^{TE} \qquad (4)$$

where j_{nm} and ν_{nm} are the roots of the Bessel function and its derivative, respectively. The result of substituting (4) into (2) are second-order differential equations for the time-dependent amplitudes *U, A, B, W, C* and *D*:

$$f(g_X)X_{nm} + b^2(2j\omega X'_{nm} - X''_{nm}) = \mathrm{F}_X(jK_X Q(t) + R_X Q'(t)),$$

$$X = U,\ A,\ B,\ W,\ C,\ D \qquad (5)$$

with

$$f(g_X) = c^2(g_X^2 + b^2k^2) - b^2\omega^2 = b^2(\widetilde{\omega}_N^2 - \omega^2),$$

$$\widetilde{\omega}_N = \sqrt{c^2\,g_X^2/b^2 + k^2}, \qquad (6)$$

$g_X = j_{nm}$ for TM modes and $g_X = \nu_{nm}$ for TE modes.

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Further in (5):

$$F_U = -q\mu_0 \frac{c^2}{2\pi^2}\frac{J_n\left(j_{nm}\frac{a}{b}\right)}{J_{n+1}^2(j_{nm})},$$

$$F_A = -jq\mu_0 \frac{c^2}{2\pi^2}\frac{J_n(j_{nm}\,a/b)}{j_{nm}^2 J_{n+1}^2(j_{nm})},$$

$$F_B = -q\mu_0 \frac{bac^2\omega_b}{2\pi^2}\frac{\nu_{nm}}{\nu_{nm}^2-n^2}\frac{J_n'(\nu_{nm}\,a/b)}{J_n^2(\nu_{nm})}$$

$$K_U = c^2k - V\omega, \qquad R_U = V$$

$$K_A = c^2 j_{nm}^2 - b^2 n\omega_b\omega, \qquad R_A = b^2 n\omega_b$$

$$K_B = \omega, \qquad R_B = -1$$

$$F_W = j\, q\frac{2ac\omega_b}{b\varepsilon_0}\frac{\nu_{nm}^3}{\nu_{nm}^2-n^2}\frac{J_n'(\nu_{nm}\,a/b)}{J_n^2(\nu_{nm})},$$

$$K_w = 1, \qquad R_W = 0,$$

$$F_C = jq\frac{2c}{\varepsilon_0}\frac{j_{nm}^2 V - b^2 kn\omega_b}{J_{n+1}^2(j_{nm})}\frac{J_n(j_{nm}\,a/b)}{j_{nm}^2},$$

$$K_C = 1, \qquad R_C = 0,$$

$$F_D = -q\frac{2abck\omega_b}{\varepsilon_0}\frac{\nu_{nm}}{\nu_{nm}^2-n^2}\frac{J_n'(\nu_{nm}\,a/b)}{J_n^2(\nu_{nm})},$$

$$K_D = 1, \qquad R_D = 0 \qquad (7)$$

Equation (5) is a second order differential equation. Its complete solution can be composed of a particular solution of an inhomogeneous equation and a general solution of a homogeneous equation (with zero right-hand side). The solution of the inhomogeneous equation (5) can be obtained by representing the amplitudes X_{nm} and function *Q(t)* through the images $\tilde{X}_{nm}$ and $\tilde{Q}_{nm}$ of the Laplace transform in time:

$$X_{nm} = \int_0^\infty \tilde{X}_{nm}e^{-\alpha t}d\alpha\,, \quad Q(t) = \int_0^\infty \tilde{Q}e^{-\alpha t}d\alpha \qquad (8)$$

After substitution one obtains:

$$\tilde{X}_{nm}(\alpha) = F_X\tilde{Q}(\alpha)P_X(\alpha),$$

$$P_X(\alpha) = jK_X - \alpha R_X/f(g_X) - b^2(2j\omega\alpha + \alpha^2) \qquad (9)$$

The time dependence of the amplitude is determined using the inverse Laplace transform from the coordinate α to the time domain. For all six components, the solution has an identical form:

$$X(t) = F_X\mathcal{L}_t^{-1}\{\tilde{Q}(\alpha)P_X(\alpha)\},\ X = U,\ A,\ B,\ W,\ C,\ D \quad (10)$$

An explicit solution can be obtained from (10) by substitution of the Laplace image $\tilde{Q}(\alpha)$ by a specific function $Q(t)$ of the time dependent function of a charge variation.

RADIATION OF INSERTED PARTICLE

Now consider the process of injection of a single point particle emerging at the time instant $t = 0$ at the point $r = a, z = 0, \varphi = 0$ inside the waveguide and being drawn into motion along a helical trajectory by external magnetic fields. The phenomenon of the instantaneous appearance of a charged point particle is described by the introduction of a step function into the expressions (1) for the charges and currents instead of a function $Q(t)$: $Q(t) = \chi(t)$, where $\chi(t) = 0$ at $t < 0$ and $\chi(t) = 1$ at $t \geq 0$. The Laplace image of a step function is:

$$\tilde{\chi}(\alpha) = \mathcal{L}_\alpha\{\chi(t)\} = \alpha^{-1} \qquad (11)$$

and the derivative of the step function at $t > 0$ is zero, just as $R_X = 0$ in (9). Therefore, from (10) we have:

$$X_N(t) = j\frac{F_N K_N}{f(g_N)}\left\{1 - e^{-j\omega t}\left(cos(\tilde{\omega}_N t) + j\frac{\omega}{\tilde{\omega}_N}sin(\tilde{\omega}_N t)\right)\right\} \qquad (12)$$

The first term in (12) coincides with the expression for the stationary solution in an infinite waveguide.

The factor $u_0 = b^2/f(g_N)$ can be represented as a sum of two terms:

$$u_0 = \frac{1}{(k_1-k_2)(k-k_1)} - \frac{1}{(k_1-k_2)(k-k_2)}. \qquad (13)$$

On the other hand, $u_0 = u_1 + u_2$ with

$$u_1 = \frac{k}{k_1(k_1-k_2)(k-k_1)} - \frac{k}{k_2(k_1-k_2)(k-k_2)}$$

$$u_2 = \frac{1}{k_2(k_1-k_2)} - \frac{1}{k_1(k_1-k_2)} = \frac{1}{k_1k_2} \qquad (14)$$

where

$$k_{1,2} = \frac{bVn\omega_0 \pm c\sqrt{b^2n^2\omega_0^2 - g_N^2(c^2-V^2)}}{b(c^2-V^2)} \qquad (15)$$

are the roots of equation $f(g_N) = 0$ with respect to k. Now (12) can be rewritten as:

$$X_N(t) = jF_N K_N u_0 - e^{-j\omega t}Z(k)u_1$$
$$-e^{-j\omega t}Z(k)u_2 + e^{-j\omega t}Z_0(k) \qquad (16)$$

with

$$Z(k) = jF_N K_N\left\{\cos(\tilde{\omega}_N t) + j\frac{\omega}{\tilde{\omega}_N}\sin(\tilde{\omega}_N t)\right\}$$

and

$$Z_0(k) = A_0(k)\cos(\tilde{\omega}_N t) + B_0(k)\sin(\tilde{\omega}_N t) \qquad (17)$$

where (17) is a general solution of the homogeneous equation for the amplitudes (with zero right-hand side), which is given by

$$G_X = f(g_X)X_{nm} + b^2(2j\omega X_{nm}' - X_{nm}'') = 0 \qquad (18)$$

with the so far undefined coefficients $A_0(k)$ and $B_0(k)$.

The transition to the space-time domain is accomplished using the inverse Fourier transform versus k:

$$X_N(t) = X_N^{(0)}(t) + X_N^{(1)}(t) + X_N^{(2)}(t) + X_N^{(3)}(t) \qquad (19)$$

$$X_N^{(0)}(t) = jF_N\int_{-\infty}^{\infty} K_N u_0 e^{j(\omega t - kz)}dk,$$

$$X_N^{(1)}(t) = \int_{-\infty}^{\infty} Z(k)u_1 e^{-jkz}dk$$

$$X_N^{(2)}(t) = \int_{-\infty}^{\infty} Z(k)u_2 e^{-jkz}dk,$$

$$X_N^{(3)}(t) = \int_{-\infty}^{\infty} Z_0(k)e^{-jkz}dk \qquad (20)$$

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The integrand of $X_N^{(0)}(t)$ is an analytic function on the entire complex plane k and, with the exception of the points $k = k_1$ and $k = k_2$, corresponding to two simple poles lying on the real axis. Its value (in the sense of the principal value) is determined by the residues of these poles in the usual way [5]. The integrand of $X_N^{(1)}(t)$ has the same poles, but it is not an analytic function. The result of the integration along the edges of the cuts emanating from the branch points $k = \pm j\, g_N/b$ should also be added to its value as the contributions from the poles. Asymptotically this contribution can be calculated by the saddle point or stationary phase method [12]. It is easy to see, that the contribution from the poles to the integral $X_N^{(1)}(t)$ completely compensates the integral $X_N^{(0)}(t)$, and that the additional contribution of the integral $X_N^{(1)}(t)$ (calculated, for example, by the stationary phase method), are compensated by the appropriate selection of the amplitudes $A_0(k)$ and $B_0(k)$ in the integral $X_N^{(3)}(t)$. Only the third term $X_N^{(2)}(t)$ in (20) remains nonzero. It does not contain poles, but its integrand is not an analytic function either. This integral

$$X_N^{(2)}(t) = \frac{j\mathrm{F}_U}{k_1 k_2}\int_{-\infty}^{\infty} K_U\{\cos(\widetilde{\omega}_N t) + j\frac{\omega}{\widetilde{\omega}_N}\sin(\widetilde{\omega}_N t)\}\, e^{-jkz} dk \qquad (21)$$

can be calculated explicitly. In particular, for the longitudinal electric component E_z we have:

$$E_z = jq\frac{\pi}{2c^2}\frac{F_U}{k_1 k_2} J_n\left(f\frac{r}{c}\right) e^{jn\varphi} S(y,t) \qquad (22)$$

Here

$$S(y,t) = AJ_1(fu) - j[BJ_2(fu) - CJ_0(fu)],$$

$$A = \frac{fn\omega_b}{u}(Vct + c^2 y - 2V^2 y),$$

$$B = \frac{f^2(c^2-V^2)(Vt^2+2yct+Vy^2)}{2cu^2},$$

$$C = \frac{V(f^2(c^2-V^2)+2c^2n^2\omega_b^2)}{2c},$$

$$u = \sqrt{t^2-y^2},\; y = z/c,\; f = j_{nm}c/b \qquad (23)$$

and $J_l(x)$ $(l = 0,\ 1,\ 2,\ n)$ are Bessel functions of the first kind.

For the calculation the following relation [13] was used

$$\int_0^{\infty}\frac{\sin\left[t\sqrt{f^2+x^2}\right]}{\sqrt{f^2+x^2}}\cos(xy)\,dx = \frac{\pi}{2}\begin{cases} J_0\left[f\sqrt{t^2-y^2}\right], & 0<y<b \\ 0, & b<y<\infty \end{cases} \qquad (24)$$

In contrast to the case of the homogeneous motion in an infinite waveguide, the waveguide is filled with energy as the particle travels into the waveguide in the case under consideration here. The front of the propagated wave is determined by the equality $ct = z$. Note that the field component (22) tends to a finite limit at $ct \to z$.

For the presence of radiation at a certain observation point r, φ, z inside the waveguide, the principle of causality must be observed, which requires the following relations to be satisfied:

$$\frac{l_1}{V} + \frac{l_2}{c} = t, \qquad l_1 + l_2\cos\alpha = z \qquad (25)$$

Here l_1 is the distance along the z axis, indicating the position of the particle at a certain moment of time $t' < t$, where t is the time at which its radiation reaches the point of observation r, φ, z. l_2 denotes the distance between the particle and the point of observation, α is the angle between the line connecting the particle and the observation point and the axis of the waveguide.

From (25) it follows:

$$l_1 = \frac{V(ct\cos\alpha - z)}{c\,\cos\alpha - V}, \quad l_2 = \frac{c(z-tV)}{c\,\cos\alpha - V}\;. \qquad (26)$$

In turn, from (26) it follows: If $l_{1,2} > 0$

i) the forward radiation is concentrated in the region $ct > z > tV$,

ii) the radiation is concentrated near the axis of the waveguide within the conical angle

$$\alpha \le min\left\{\sqrt{2/(1-V^2/c^2)}, \sqrt{1-(z/ct)^2}\right\} \qquad (27)$$

The frequency characteristics of the radiation can be determined by analysing the integrand in formula (21). For an arbitrary value of the function $f(j_{nm})$, it is a rapidly oscillating function, while with $f(j_{nm}) = 0$ the oscillations remain only in the phase and its modulus slowly varies with frequency. For this reason, its frequency distribution is characterized by sharp peaks at frequencies determined by equation $f(j_{nm}) = 0$. Thus, the resonant frequencies remain the same as in the stationary motion of a particle in an infinite waveguide [7].

The derived formula (23) describes a strongly directed and narrow-band radiation.

The process of the emergence and subsequent propagation of an arbitrary bunch of length t_0 can be described by a convolution of the expression for the field of a point particle (23) with the longitudinal distribution function $f_z(t)$ in the bunch:

$$E_z^b(t) = \int_0^{t_u} f_z(t-t')E_z(t')dt' \qquad (28)$$

Here $t_u = t$ for $t \le t_0$ and $t_u = t_0$ at $t > t_0$.

For brevity, we derived an explicit expression only for the longitudinal electrical component (23). The rest of the components can be calculated similarly using equations (7).

CONCLUSION

The results of this work make it possible to study the processes of emission of bunches in a helical undulator combined with a waveguide in all details, as they occur during injection, subsequent propagation, and after leaving the open end of the waveguide. They will contribute to the creation of mathematical models of the operation of an undulator-waveguide structure close to reality.

The results related to the time-varying charge can also find application in case of particle loss due to scattering on the walls of the waveguide and scattering on molecules of the residual gas.

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