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SUPPRESSION OF EMITTANCE GROWTH BY A COLLECTIVE FORCE: VAN KAMPEN APPROACH

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Abstract

In hadron synchrotrons, external sources of noise affecting the beam results in emittance growth through the mechanism of decoherence. Active feedbacks are often used to suppress this emittance growth. In the presence of beam-beam interactions, it was shown that coherent modes of oscillations with frequencies shifted outside of the incoherent spectrum significantly enhances the efficiency of the emittance growth suppression by active feedbacks. We show that the same enhancement of the emittance growth suppression may be driven by a beam coupling impedance generating a real tune shift larger than the detuning.

INTRODUCTION

We aim at quantifying the emittance growth due to an external source of noise in the presence of a frequency spread and a collective force. In the next section we describe the mathematical model based on a perturbation of the linearised Vlasov equation and obtain a basis of functions representing the modes of oscillation of the beam. In the third section the problem is reduced to an initial condition problem by assuming that the noise can be represented by a superposition of individual kicks without interference. Thus, an initial condition corresponding to an offset beam is expressed in terms of modes of oscillation, such that the time evolution of the perturbation can be written explicitly. In the fourth section, an expression for the final emittance after decoherence is obtained by considering the limit of long time scale.

MODEL

In the following we use the action-angle variables J and θ relating to the transverse position x and momentum p_x:

$$x = \sqrt{2J}\cos(\theta) \tag{1}$$
$$p_x = \sqrt{2J}\sin(\theta). \tag{2}$$

The unperturbed distribution can then be written as

$$\Psi_0(J,\theta) = \frac{1}{2\pi} f_0(J). \tag{3}$$

The effective Hamiltonian of the lattice H_0 is the one of an oscillator featuring amplitude detuning as

$$\omega(J) \equiv \frac{\partial H_0}{\partial J}. \tag{4}$$

We add a collective force proportional to the average position of the bunch

$$F_c = -2\Delta\Omega_{ext}\langle x\rangle, \tag{5}$$

with $\Delta\Omega_{ext}$ the corresponding complex frequency shift. Thus we can write the Vlasov equation as:

$$\frac{\partial\Psi_1}{\partial t} + \frac{\partial\Psi_1}{\partial\theta}\omega(J) - \frac{\partial\Psi_0}{\partial J}\sqrt{2J}\sin(\theta)F_c = 0. \tag{6}$$

We will be looking for harmonic solutions with the form

$$\Psi_1(J,\theta,t) = \frac{1}{2\pi} g(J) e^{i(\theta-\Omega t)}. \tag{7}$$

The Vlasov equation becomes:

$$(\Omega-\omega)g = \frac{-1}{2}\Delta\Omega_{ext}\frac{df_0}{dJ}\sqrt{2J}\int dJ\sqrt{2J}g. \tag{8}$$

VAN KAMPEN MODES

Coherent Mode

A solution of Eq. (8), corresponding to a coherent mode of oscillation, can be expressed as:

$$g_c = \frac{-1}{2}\Delta\Omega_{ext}\frac{\sqrt{2J}\frac{df_0}{dJ}}{\Omega_c-\omega}, \tag{9}$$

choosing the mode frequency Ω_c such that:

$$\int dJ\sqrt{2J}g_c = 1. \tag{10}$$

This condition translates into the well known dispersion relation [1]:

$$\int dJ\frac{J\frac{df_0}{dJ}}{\Omega_c-\omega} = \frac{-1}{\Delta\Omega_{ext}} \tag{11}$$

Incoherent Spectrum

Following Van Kampen [2], we find another set of solutions in the realm of distribution functions:

$$g_k = \frac{-1}{2}\Delta\Omega_{ext}\left(\frac{\sqrt{2J}\frac{df_0}{dJ}}{\Omega_k-\omega}\right)_{\text{p.v.}} + \lambda_k\delta(J-k), \tag{12}$$

with $k\in[0,\infty[$. The notation $(\cdot)_{\text{p.v.}}$ indicates that the integration of the distribution function should be performed as a Cauchy principal value. As for the coherent mode, λ_k is chosen such that

$$\int dJ\sqrt{2J}g_k = 1. \tag{13}$$

This condition yields:

$$\lambda_k = \frac{1}{\sqrt{2k}}\left(1+\Delta\Omega_{ext}\text{p.v.}\int dJ\frac{J\frac{df_0}{dJ}}{\Omega_k-\omega}\right). \tag{14}$$

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By introducing this expression into the Vlasov equation, we determine the oscillation frequencies of the incoherent modes $\Omega_k = \omega(k)$. Once the coefficients a_c and a_k are determined based on the initial condition, the time evolution of the perturbation is given by:

$$\begin{aligned}\Psi_1(J,\theta,t) &= \frac{a_c}{2\pi} g_c(J) e^{i(\theta-\Omega_c)t} \\ &+ \int dk \frac{a_k}{2\pi} g_k(J) e^{i(\theta-\Omega_k)t}. \end{aligned} \tag{15}$$

DECOMPOSITION OF THE INITIAL KICK

In order to estimate the emittance growth generated by a kick to the beam, we want to express an initial condition with a beam offset by δ_{ext} in terms of Van Kampen modes:

$$\Psi_1(J,\theta,0) = \delta_{ext}\sqrt{2J}\frac{d\Psi_0}{dJ}e^{i\theta} \tag{16}$$

$$\langle g_n, g_m\rangle \equiv \int dJ \frac{g_n g_m^*}{\frac{df_0}{dJ}}. \tag{17}$$

It can be shown that the Van Kampen modes are orthogonal, thus we can write:

$$a_c = \frac{\delta_{ext}}{|\langle g_c, g_c\rangle|},\; a_k = \frac{\delta_{ext}}{|\langle g_k, g_k\rangle|}, \tag{18}$$

with

$$\langle g_c, g_c\rangle = \frac{1}{2}|\Delta\Omega_{ext}|^2 \int dJ \frac{J\frac{df_0}{dJ}}{|\Omega_c - \omega|^2} \tag{19}$$

for the coherent mode and

$$\begin{aligned}\langle g_k, g_k\rangle &= \frac{1}{2k\frac{df_0}{dk}}\left(\pi^2\left|\frac{\Delta\Omega_{ext}}{\frac{\partial\omega}{\partial J}}\right|^2 k^2\left(\frac{df_0}{dk}\right)^2\right. \\ &+ 2\pi k\frac{df_0}{dk}\frac{\mathrm{Im}\,\Delta\Omega_{ext}}{|\frac{\partial\omega}{\partial k}|} \\ &+ \left.\left|1+\Delta\Omega_{ext}\mathrm{p.v.}\int dJ\frac{J\frac{df_0}{dJ}}{\Omega_k-\omega}\right|^2\right) \end{aligned} \tag{20}$$

for the incoherent spectrum. A key step in the derivation of this expression is the usage of the Poincaré-Bertrand formula to modify the order of integration, as in [3]. This leads to the first term in π^2 inside the parenthesis. We note also that a single pole was assumed, imposing that $\omega(J)$ is monotonic, but not necessarily linear at this point.

EMITTANCE GROWTH

Computing the time evolution of the emittance by averaging Eq. (15), we find that it is constant, indicating that the emittance growth due to a kick is a second order effect. Following [4], we obtain the second order term using Hamilton's equation for the action J and insert it into the Vlasov equation:

$$\frac{dJ}{dt} = \frac{1}{\frac{\partial\Psi_0}{\partial J}}\left(\frac{\partial\Psi_1}{\partial t} + \frac{\partial\Psi_1}{\partial\theta}\omega(J)\right). \tag{21}$$

Averaging over J and realising that, up to second order, the partial derivatives can be expressed as a total time derivative we obtain

$$\frac{d}{dt}\langle J\rangle_{\Psi_1(t)} = \frac{1}{2}\frac{d}{dt}\int dJ d\theta \frac{1}{\frac{\partial\Psi_0}{\partial J}}\Psi_1^2. \tag{22}$$

Using the expression of the time evolution of the distribution in terms of Van Kampen modes (Eq. (15)), we get

$$\begin{aligned}\langle J\rangle_{\Psi_1(t)} &= \frac{\delta_{ext}^2}{8\pi^2}\int dJ d\theta\frac{1}{\frac{\partial\Psi_0}{\partial J}} \\ &\left(\frac{1}{|\langle g_c,g_c\rangle|}g_c(J)e^{i(\theta-\Omega_c t)}\right. \\ &+ \left.\int dk\frac{1}{|\langle g_k,g_k\rangle|}g_k(J)e^{i(\theta-\Omega_k t)}\right)^2. \end{aligned} \tag{23}$$

As we are interested in the total emittance growth due to a single kick, we thus compute the limit of the average emittance towards infinite time. The limit exists if we assume that there exists a damping force, i.e. $\mathrm{Im}(\Omega_c) < 0.0$:

$$\begin{aligned}\Delta\epsilon &\equiv \lim_{t\to\infty}\langle J\rangle_{\Psi_1(t)} - \epsilon_0 \\ &= \frac{1}{2}\delta_{ext}^2\int\frac{dk}{|\langle g_k,g_k\rangle|}, \end{aligned} \tag{24}$$

where we have introduced the unperturbed r.m.s. emittance $\langle J\rangle_{\Psi_0} \equiv \epsilon_0$. The contribution of the coherent mode a_c vanishes when considering the limit, showing that all the energy imparted to the collective motion is removed by the damping force and thus does not contribute to the emittance growth. Only the incoherent spectrum contributes to the emittance growth. This result is consistent with the ones obtained in the frame of coherent beam-beam interactions [4]. In absence of collective force, the emittance growth due to a kick is given by $\frac{1}{2}\delta_{ext}^2$ [5], thus it is convenient to define η, the emittance growth factor due to the collective force:

$$\eta \equiv \int\frac{dk}{|\langle g_k,g_k\rangle|} \tag{25}$$

which can be solved numerically using Eq. (20).

Gaussian Distribution and Linear Detuning

We apply the results above to a practical configuration featuring linear detuning and a Gaussian distribution of the particles. The lattice Hamiltonian and corresponding oscillation frequencies are

$$H_0 = \omega_0\left(Q_0 J + \frac{a}{2}J^2\right) \text{ and } \omega(J) = \omega_0(Q_0 + aJ), \tag{26}$$

with ω_0 being the revolution frequency, Q_0 the unperturbed tune and a the linear detuning coefficient. The unperturbed particle distribution is

$$f_0 = \frac{1}{\epsilon_0}e^{-J/\epsilon_0}. \tag{27}$$

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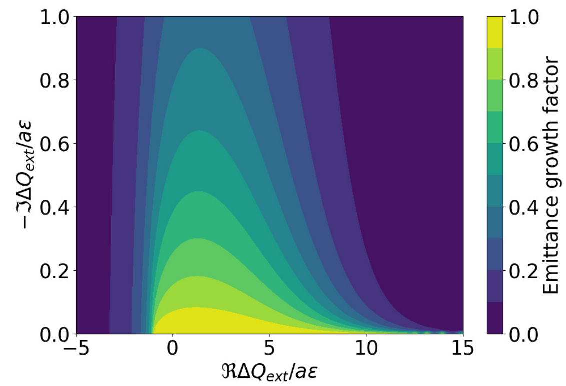

Figure 1: Emittance growth factor with a positive detuning coefficient for different real and imaginary part of the external frequency shift.

We have

$$\eta = \int dI \frac{Ie^{-I}}{G(I)} \tag{28}$$

with

$$\begin{aligned} G(I) &= \left| \left| 1 + \frac{\Delta\Omega_{ext}}{a\omega_0\epsilon_0}\left(1 + Ie^{-I}E_1(-I)\right)\right|^2 \right. & (29) \\ &- 2\pi Ie^{-I}\frac{\operatorname{Im}\Delta\Omega_{ext}}{|a\omega_0\epsilon_0|} & (30) \\ &+ \left. \pi^2\left|\frac{\Delta\Omega_{ext}}{a\omega_0\epsilon_0}\right|^2 I^2e^{-2I}\right|. & (31) \end{aligned}$$

E_1 is the exponential integral. We find that the emittance growth factor is entirely determined by the external complex frequency shift scaled by the r.m.s. frequency spread $\frac{\Delta\Omega_{ext}}{\omega_0 a\epsilon_0}$. This dependence is shown in Fig. 1. The presence of a damping component reduces the emittance growth in all configurations. This behaviour corresponds to the expectation for a resistive feedback. In addition we observe that a real frequency shift, e.g. by introducing a reactive component to the feedback system, can significantly reduce the emittance growth if the real frequency shift is significantly larger than the frequency spread. There is a strong asymmetry between negative and positive real frequency shifts. In this example the detuning coefficient a is positive, such that a negative coherent frequency shift is favourable to suppress the emittance growth. This can be understood with Fig. 2. The incoherent spectrum is mostly excited for positive real external frequency shifts, i.e. external frequency shifts matching the oscillation frequencies of individual particles. In this regime, Eq. (11) does not admit any solution for Ω_c, implying that the coherent mode does not exist, only incoherent ones. Since only the energy imparted to the incoherent modes leads to emittance growth, it explains why the emittance growth suppression is least effective in this regime. For larger real external frequency shifts with either sign, the coherent mode dominates the dynamics and thus

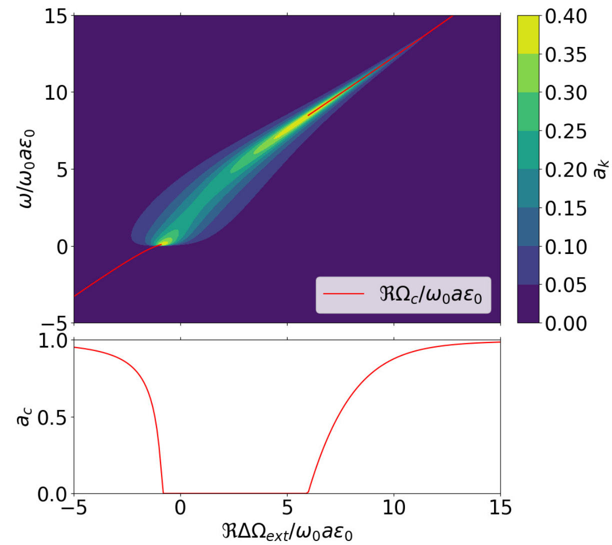

Figure 2: Spectrum of oscillation resulting from a kick for different external frequency shifts Re ΔQ_{ext} (top plot). The imaginary part is kept constant at $2 \cdot 10^{-4}$ and the detuning $a\epsilon_0$ is 10^{-3}. The incoherent contribution is shown with a colour scale, whereas the coherent mode is represented by a red line, the corresponding coefficient is shown on the bottom plot.

the emittance growth is effectively suppressed.

CONCLUSION

Using the Van Kampen mode approach, it is shown that a collective force can significantly reduce the emittance growth due to an external source of noise. A damping force is needed to obtain a suppression of the emittance growth, yet a real frequency shift may enhance this suppression, mostly in configurations where the real frequency shift is larger than the frequency spread, leading to the emergence of a coherent mode.

This behaviour is analogous to the one obtained in the frame of beam-beam interactions [4], namely the existence of a coherent mode outside of the incoherent spectrum leads to an effective suppression of the emittance growth. This work shows that the presence of beam-beam coherent modes is not necessary to achieve the suppression, but it can be achieved for example with an active feedback featuring both a resistive and a reactive component. It may also occur due to a complex frequency shift driven by the machine impedance, as shown in simulations in [6, 7].

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REFERENCES

[1] J. Scott Berg and F. Ruggiero, "Landau damping with two-dimensional betatron tune spread", CERN, Geneva, Switzerland, rep. SL-AP-96-71-AP, Dec. 1996 https://cds.cern.ch/record/318826

[2] N.G. Van Kampen, "On the Theory of Stationary Waves in Plasmas", *Physica*, vol. 21, pp. 949–963, 1955. doi:10.1016/S0031-8914(55)93068-8

[3] N. I. Muskhelishvili,"Singular Integral Equations", Gronigen, Netherlands:Wolters-Noordhoff Publishing, 1958, pp. 57. doi:10.1007/978-94-009-9994-7

[4] Y. Alexahin, "A study of the coherent beam–beam effect in the framework of the Vlasov perturbation theory", *Nucl. Instrum. Methods Phys. Res. A*, vol. 480, pp. 253–288, 2002. doi:10.1016/S0168-9002(01)01219-0

[5] D. Möhl,"Sources of emittance growth", in *CAS - CERN Accelerator School: Intermediate Accelerator Physics*, Geneva, Switzerland: CERN, pp. 245–270. doi:10.5170/CERN-2006-002.245

[6] N. Triantafyllou *et al.*, "Investigation of Damping Effects of the Crab Cavity Noise Induced Emittance Growth", in *Proc. IPAC'21*, Campinas, Brazil, May 2021, pp. 2054–2057. doi:10.18429/JACoW-IPAC2021-TUPAB256

[7] N. Triantafyllou *et al.*, "Investigating the Suppression of the Crab Cavity Noise Induced Emittance Growth From the Transverse Beam Impedance", presented at IPAC'22, Bangkok, Thailand, June 2022, paper WEOZSP2, this conference.