DISCUSSION OF THE OPTIMISATION OF A LINAC LATTICE TO MINIMISE DISRUPTION BY A CLASS OF PARASITIC MODES

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Abstract

It is well known that each resonant mode in the RF spectrum of multi-cell accelerating cavities will split into a passband containing a number of modes, and that the coupling of these modes to the beam is dependent on the velocity of the accelerated particles. If these modes are found to degrade the quality of the beam, it is possible to take various measures to damp them, and thus keep their effect below some critical threshold. In the case of the parasitic modes within the same passband as the fundamental accelerating mode, their frequency is typically too close to that of the fundamental to allow their power to be safely extracted, and so cavity designers must rely on the natural damping of the cavity itself. This note contains a theoretical discussion of the coupling of the beam to these passband modes for a large class of accelerating cavities, and provides a mathematical model for use during the design and optimisation of linacs.

INTRODUCTION

In general, π-mode accelerating cavities act as several coupled cells, each of which resonates at almost the same frequency. The cell-cell coupling splits each of the resonances into a passband containing the same number of modes as there are coupled cells. Each of these will have the same character (i.e. TM_{mnp}, TE_{mnp}, etc.) as the single cell resonance, but may be differentiated by their frequency and cell-cell phase difference.

Modes lying within the same passband as the accelerating mode will be referred to as Same Order Modes (SOMs), while others will be called Higher Order Modes (HOMs).

Various schemes are used to damp the power in these parasitic modes so that their amplitude has fallen below some threshold value by the time a subsequent beam pulse arrives at the cavity, however the SOMs remain problematic since the similarity of their frequency and structure with the desired fundamental mode mean that it is normally not possible to damp their power sufficiently.

In storage rings or CW linacs, the pulses last long enough that modes whose frequency, f_m, does not lie within a very small region around an integer multiple of the bunch repetition frequency, $n \cdot f_b \pm \Delta f$, will be "washed out", and will not obtain an amplitude large enough to cause problems. In other words, these modes will be excited at all phases, thus resulting in almost complete cancellation.

In the case of pulsed machines, the shortness of the pulse

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will increase the window, Δf, in which damaging parasitic modes might exist. Equivalently, decreasing the pulse length will increase the proportion of the bunch train that will experience high field amplitudes for the parasitic modes before their phase slip with respect to the bunch frequency begins to damp them. Therefore, studies of these modes are important for pulsed machines such as the European Spallation Source (ESS).

Velocity range

In machines that accelerate "heavy" particles such as protons or ions, the accelerating cavities must be designed so as to handle the changing velocity of the beam. Often, linacs are divided into various families of cavities, each of which is optimised for a particular beam velocity. These cavities will have a velocity range over which the efficiency of the acceleration is considered to be acceptable, and so optimisation of the linac design proceeds while taking these boundaries into account.

Just as the efficiency of the accelerating mode (i.e. the coupling of the mode to the beam) is a function of the beam velocity, so is the coupling to the SOMs, and there may be a velocity range where the coupling to a SOM exceeds that of the accelerating mode. In this case, it is likely that this mode will cause significant deterioration of the quality of the beam pulse.

Therefore, the function of the beam's coupling to the SOMs may provide a tighter limit on the acceptable velocity range of the cavity than would otherwise be expected.

CAVITY MODEL

The cavity model used here follows the derivation used in [1], although note that there are several typos in that paper that make the quantitative conclusions unreliable. Much of the calculation may also be extrapolated from [2].
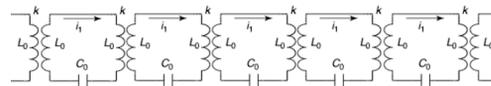

Figure 1: Lumped circuit model of a a five-cell cavity connected at both ends to a beam-pipe.

Figure 1 shows the lumped circuit used to model a five-cell cavity coupled to a beam-pipe.

The coupling, k, between each cell, and between the end cells and the beam pipe, is modeled as a transformer, and the cells are modelled as resonant circuits. The values of k, L_0, & C_0, may be derived from the desired response of the circuit (i.e. resonant frequency, etc.).

Note that the model shown in figure 1 is appropriate for a cavity composed of identical cells. The standard presentation of this circuit removes the transformers from either end cell, each of which are terminated with a short. The halving of the cell's inductance is compensated by doubling the value of the capacitance in each of these end cells. This results in a string of cells with the correct amplitude variation, however, as noted in [1], the phase relationship between the cells is no longer modelled correctly.

A more correct approach would be to start with the model shown in figure 1, but add an error, δf, to the resonant frequency of the final cells. This error will alter the frequency & phase characteristics of each of the modes in the passband, and can be set so as to achieve the desired field characteristics for the accelerating mode.

BEAM-CAVITY COUPLING

The purpose of this note is to determine the functional relationship between the beam's velocity, $v = \beta c$, and its coupling to the fundamental mode & the nearest SOM. The permissible velocity range of the cavity is then arbitrarily defined as the points on either side of this maximum where the couplings to the two modes has become equal.

Cell amplitudes

In the model presented here, each cell has the same frequency, $\omega_0 = 2\pi f_0 = 1/\sqrt{2L_0 C_0}$; i.e. the frequency error usually introduced to tune the field is not present.

For each cell, an equation can be written by summing the currents around each circuit.

$$X_1 \left[1 - \frac{\omega_0^2}{\omega_q^2} \right] + X_2 \frac{k}{2} = 0 \quad n = 1 \quad (1)$$

$$X_n \left[1 - \frac{\omega_0^2}{\omega_q^2} \right] + (X_{n-1} + X_{n+1}) \frac{k}{2} = 0 \quad 2 \geq n \geq N - 1 \quad (2)$$

$$X_N \left[1 - \frac{\omega_0^2}{\omega_q^2} \right] + X_{N-1} \frac{k}{2} = 0 \quad n = N \quad (3)$$

These equations describe the current, $X_n \equiv i_k \sqrt{2L_n}$, in the k-th cell of the cavity, oscillating in the q-th mode with an angular frequency, ω_q. They may be rearranged as a matrix equation as follows.

$$LX = AX \quad (4)$$

$$L \equiv \omega_0^{-2} \begin{bmatrix} 1 & \frac{k}{2} & & \cdots & 0 \\ \frac{k}{2} & 1 & \frac{k}{2} & & \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \frac{k}{2} & 1 & \frac{k}{2} \\ 0 & \cdots & & \frac{k}{2} & 1 \end{bmatrix} \quad (N \times N) \quad (5)$$

$$X \equiv \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix} \quad (N \times 1) \quad (6)$$

$$A \equiv \omega_q^{-2} \quad (7)$$

Note that equation 4 has the same form as an eigenvalue equation, and so the vector, X, is an eigenvector of the matrix, L, while the scalar, A, is an eigenvalue.

Therefore, the frequencies & distribution of the cell amplitudes can be found for each of the N modes by finding the eigenvalues/vectors of the $N \times N$ matrix, L.

For a general mode, q, it can be shown that the j-th component of the eigenvector, $X_{q,j}$, is as follows.

$$X_{q,n} = K_q \cdot \sin\left(\pi n \left(1 - \frac{q}{N+1} \right) \right) \quad q, n = 1, \ldots, N \quad (8)$$

Here K_q is an arbitrary scaling constant that may be used to normalize the eigenvector to be of unit length.
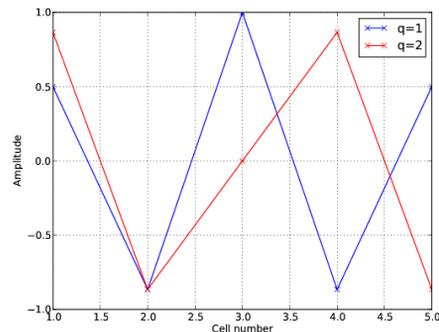

Figure 2: Amplitude of the accelerating mode, $q = 1$, & the nearest SOM, $q = 2$, in each cell of a five-cell cavity. Note that the scaling constants have been set, $K_1 = K_2 = 1$.

Figure 2 shows the expected eigenvector amplitudes for the accelerating mode, $q = 1$, & the nearest SOM, $q = 2$, of a five-cell cavity. Note that the accelerating mode is significantly different from the usual flat field-profile due to the fact that each of the cells in this calculation are modelled as having the same resonant frequency. This is due to the fact that the usual shift in frequency of the end-cells has not been taken into account.

In the remainder of this note, it will be assumed that this tuning has occurred, and that the amplitudes of the accelerating mode in each cell are identical. This assumption will not be used for the remaining modes, thereby introducing a small error into the results.

The symmetry of the nearest SOM around the central cell of the cavity can be seen quite clearly, as well as the fact that this mode has an odd distribution; i.e. a negative parity under reflection about the centre cell.

Field profile

In order to simplify subsequent calculations, the field profile of the beam is modelled as a concatenation of a series of half-cycles of sine-waves (one for each cell) whose amplitude is determined by equation 4. Figure 3 shows the
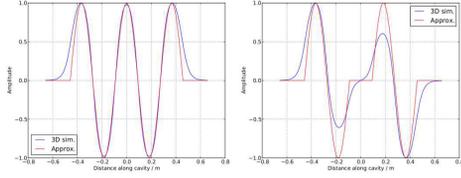
Figure 3: Approx. profile of the accelerating mode in a 5 cell cavity compared with a 3D simulation.

results of this in comparison to data extracted from a 3D simulation of a similar cavity.

These figures show that this approximation is sufficient for the accelerating mode – although remember that these amplitudes are set to be constant, and not gathered from equation 4. The approximation for the SOM is not so good, however it can be argued that the locations where the approximation over-estimates the field are cancelled by those where it is under-estimated.

Integration with beam velocity

This approximation to the field profile can then be integrated with a term that provides the necessary phase variation of the RF oscillating with frequency, $\omega = 2\pi f$.

The following determines the voltage, $V_{q,n}$, experienced by a particle moving with velocity, $v = \beta c$, along the z axis of cell, n. Note that the limits of the integral are defined as the beginning & end of a cell designed for a particle moving with speed, $v_0 = \beta_0 c$.

$$V_{q,n} = X_{q,n} \int_{z_n}^{z_{n+1}} \sin\left(\frac{\omega_q}{\beta_0 c}(z - z_n)\right) \sin\left(\frac{\omega_q}{\beta c}z\right) dz \quad (9)$$

$$z_n \equiv (2n - N)\frac{\beta_0 \lambda}{2} \quad (10)$$

This integral may be solved by integrating by parts twice in order to obtain the following.

$$V_{q,n} = X_{q,n}\frac{\beta^2}{\beta_0^2 - \beta^2}\frac{\beta_0 c}{\omega_q}\cdot$$
$$\left[\sin\left((2n-N)\frac{\beta_0}{\beta}\frac{\pi}{2}\right) + \sin\left((2n-N-2)\frac{\beta_0}{\beta}\frac{\pi}{2}\right)\right] \quad (11)$$

Note that equation 11 is completely general in that no assumptions have been made about the eigenvector, $X_{q,n}$, or about the number of cells, N.

To obtain the voltage for the entire cavity, this result is then summed over the number of resonant cells.

$$V_q = \sum_{n=1}^{N} V_{q,n} \quad (12)$$

It can be seen that symmetric structure within $X_{q,n}$, in combination with the parity property of the $\sin$ function, will lead to cancellations in equation 11, and a substantial reduction in the number of terms within the summation.

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RESULTS

The consequences of equations 11 & 12 are examined for the case of a 5-cell cavity (i.e. $N = 5$).

$$V^{(4)} = \frac{\beta^2}{\beta_0^2 - \beta^2}\frac{\beta_0 c}{\omega}\left[\sin\left(\frac{\beta_0}{\beta}\frac{5\pi}{2}\right) - \sin\left(\frac{\beta_0}{\beta}\frac{\pi}{2}\right)\right] \quad (13)$$

$$V^{(5)} = -\frac{2}{\sqrt{5}}\frac{\beta^2}{\beta_0^2 - \beta^2}\frac{\beta_0 c}{\omega}\cos\left(\frac{\beta_0}{\beta}\frac{5\pi}{2}\right) \quad (14)$$

Here the voltages, $V^{(4)}$ & $V^{(5)}$, correspond to the voltages obtained for the nearest SOM ($4\pi/5$ mode, $q = 2$), and the accelerating mode ($5\pi/5$, $q = 1$) respectively.

Note that the integral performed to obtain the result in equation 14 used a cosine variation of the field due to the even parity of the eigenvector for that mode (i.e. the second $\sin()$ in equation 9 was exchanged for a $\cos()$).
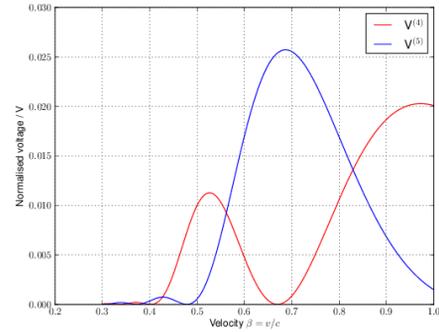

Figure 4: $V^{(4)}$ & $V^{(5)}$ vs β for a 5-cell, $\beta_0 = 0.67$, cavity.

Figure 4 shows the variation of $V^{(4)}$ and $V^{(5)}$ for a 5-cell cavity designed for $\beta_0 = 0.67$.

Note that an estimation of the acceptable velocity range based purely on the coupling to the accelerating mode might conclude with $0.54 \geq \beta \geq 0.92$, which could lead to subsequent problems due to the excessive excitation of the SOM at either end of the range.

DISCUSSION

Many computer codes used to optimise high power linacs neglect to take account of these additional modes, and so it is possible that the optimisation will lead to solutions whereby a SOM is sufficiently excited to disrupt the phase space of the beam [3].

While a general solution to the problem, $V^{(N-1)}(\beta) = V^{(N)}(\beta)$, may be difficult to write down explicitly, it would be possible to add such limits to a computer code.

REFERENCES

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