ADIABATIC THERMAL BEAM EQUILIBRIUM IN AN ALTERNATING-GRADIENT FOCUSING FIELD*

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Abstract

An adiabatic warm-fluid equilibrium theory for a thermal charged-particle beam in an alternating gradient (AG) focusing field is presented. Warm-fluid equilibrium equations are solved in the paraxial approximation and the rms beam envelope equations and the self-consistent Poisson equation, governing the beam density and potential distributions, are derived. The theory predicts that the 4D rms thermal emittance of the beam is conserved, but the 2D rms thermal emittances are not constant. Although the presented rms beam envelope equations have the same form as the previously known rms beam envelope equations, the evolution of the rms emittances in the present theory is given by analytical expressions. The beam density is calculated numerically, and although it does not have the simplest elliptical symmetry, the constant density contours are ellipses whose aspect ratio decreases as the density decreases along the transverse displacement from the beam axis. For high-intensity beams, the beam density profile is flat in the center of the beam and falls off rapidly within a few Debye lengths; and the rate at which the density falls is approximately isotropic in the transverse directions.

INTRODUCTION

A fundamental understanding of the equilibrium and stability properties of high-intensity electron and ion beams in periodic focusing fields is important in high energy density physics research [1], and in the design and operation of particle accelerators [2], such as storage rings, rf and induction linacs, and high-energy colliders. For such systems, beams of high quality (i.e., low emittance, high current, small energy spread, and low beam loss) are required. Exploration of equilibrium states of charged-particle beams and their stability properties is critical to the advancement of basic particle accelerator physics.

There exists an extensive body of literature on Vlasov beam and cold-beam equilibria [3-7] that use a δ-function phase-space distribution, which is unphysical. These theories do not take into account beam temperature effects. In addition, a formal multiple scale analysis (i.e., a third-order averaging technique) has been applied to obtain approximate Vlasov and thermal equilibria in periodic solenoidal and AG focusing fields [8]. Such an averaging procedure is valid for sufficiently small vacuum phase advances.

Recently, we have developed warm-fluid [9] and kinetic [10] equilibrium theories for an adiabatic thermal

charged-particle beam in a periodic solenoidal focusing field. The density profile of the adiabatic thermal beam equilibrium provides a more realistic representation of a laboratory beam than the uniform density profile in the KV-like beam equilibrium. Good agreement has been found between our theory and the experimental measurements [11] from the anode aperture to a distance prior to wave breaking.

In this paper, an adiabatic warm-fluid equilibrium theory of a thermal charged-particle beam in a periodic quadrupole magnetic focusing field is presented. Warm-fluid equilibrium equations are used to derive expressions for the flow velocity and beam density distribution, the rms beam envelope equations, and a self-consistent Poisson equation. A numerical technique for computing warm-fluid beam equilibria is developed and an example of the thermal beam equilibrium is presented.

WARM-FLUID BEAM EQUILIBRIUM EQUATIONS

We consider a thin, continuous, single-species charged-particle beam, propagating with constant axial velocity $V_z\hat{\mathbf{e}}_z$ through a quadrupole magnetic field. We solve the warm-fluid equilibrium equations [4] making use of the paraxial approximation. Further, we seek the solution associated with adiabatic beam propagation in a periodic quadrupole magnetic focusing. We can derive that the product of the transverse temperature and the effective transverse rms beam area is a constant, i.e.,

$$T_\perp(s)x_{brms}(s)y_{brms}(s) = const, \qquad (1)$$

where x_{brms} and y_{brms} are the rms beam envelopes. Also, we seek a solution for the equilibrium beam velocity profile of the form

$$\mathbf{V}_\perp(x,y,s) = x\frac{x'_{brms}(s)}{x_{brms}(s)}\beta_b c\hat{\mathbf{e}}_x + y\frac{y'_{brms}(s)}{y_{brms}(s)}\beta_b c\hat{\mathbf{e}}_y. \qquad (2)$$

where c is the speed of light in vacuum and $\beta_b \approx V_z/c$.

We integrate the warm-fluid momentum equation to obtain the density profile with the help of Eq. (2)

$$
\begin{aligned}
&n_b(x,y,s) \\
&= f(s)\exp\left(-\frac{\gamma_b m \beta_b^2 c^2}{2k_B T_\perp(s)}\left\{\left[\frac{x''_{brms}(s)}{x_{brms}(s)}+\kappa_q(s)\right]x^2 + \left[\frac{y''_{brms}(s)}{y_{brms}(s)}-\kappa_q(s)\right]y^2\right\}\right) \\
&\times \exp\left[-\frac{q\phi^{self}(x,y,s)}{\gamma_b^2 k_B T_\perp(s)}\right],
\end{aligned}
$$

$$(3)$$

where $f(s)$ is an arbitrary function of s to be determined later [see Eq. (9)], $\phi^{self}(x,y,s)$ satisfies the Poisson equation $\left(\partial^2/\partial x^2 + \partial^2/\partial y^2\right)\phi^{self}(x,y,s) = -4\pi q n_b(x,y,s)$, q and m are the particle charge and rest mass, respectively, γ_b is the relativistic mass factor, which, to the leading order, is $\gamma_b = const = \left(1-\beta_b^2\right)^{-1/2}$.

Using the density profile in Eq. (3), we calculate the rms beam envelopes in the x- and y- directions

$$x''_{brms}(s) + \kappa_q(s)x_{brms}(s) + \frac{K}{2qN_b}\left\langle x\frac{\partial\phi^{self}(x,y,s)}{\partial x}\right\rangle = \frac{k_B T_\perp(s)}{\gamma_b m\beta_b^2 c^2 x_{brms}(s)},$$

(4)

$$y''_{brms}(s) - \kappa_q(s)y_{brms}(s) + \frac{K}{2qN_b}\left\langle y\frac{\partial\phi^{self}(x,y,s)}{\partial y}\right\rangle = \frac{k_B T_\perp(s)}{\gamma_b m\beta_b^2 c^2 y_{brms}(s)}.$$

(5)

where $K \equiv 2N_b q^2/\gamma_b^3 m\beta_b^2 c^2$ is the self-field perveance and N_b is the number of particles per unit axial length.

To simplify the envelope equations (4) and (5), we postulate the following expressions $\left\langle x\,\partial\phi^{self}(x,y,s)/\partial x\right\rangle = -qN_b x_{brms}(s)/[x_{brms}(s)+y_{brms}(s)]$ and $\left\langle y\,\partial\phi^{self}(x,y,s)/\partial x\right\rangle = -qN_b y_{brms}(s)/[x_{brms}(s)+y_{brms}(s)]$. We numerically demonstrate that the two expressions are satisfied. In Fig. 1, we plot the percentage differences between the right-hand side and left-hand side of the two expressions versus the scaled normalized perveance $\hat{K} \equiv SK/4\varepsilon_{4Dth}$ for $x_{brms}/y_{brms}=2$. We use the calculated potential and density profiles to compute the left-hand sides, whereas the right-hand sides are evaluated analytically. As shown in Fig. 1, for a wide range of $\hat{K}$ the differences are less than 1.1%, which is small. Similar results are obtained for the ratio x_{brms}/y_{brms} ranging from 1 to 5. Therefore, we conclude that the postulated expressions are satisfied.
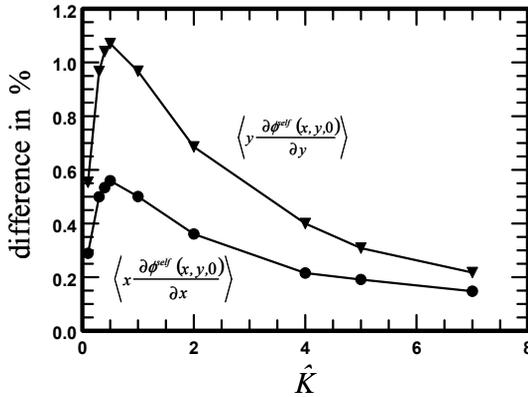

Figure 1: Plots of the percentage differences between $\left\langle x\,\partial\phi^{self}(x,y,s)/\partial x\right\rangle$ and $-qN_b x_{brms}(s)/[x_{brms}(s)+y_{brms}(s)]$ (circles) and between $\left\langle y\,\partial\phi^{self}(x,y,s)/\partial x\right\rangle$ and $-qN_b y_{brms}(s)/[x_{brms}(s)+y_{brms}(s)]$ (triangles) versus the scaled normalized perveance $\hat{K}$ for $x_{brms}/y_{brms}=2$.

We introduce the 2D rms thermal emittances of the beam, ε_{xth} and ε_{yth}, defined by

$$\varepsilon_{xth}^2(s) \equiv (\beta_b c)^{-2}\left\langle x^2\right\rangle_\Gamma \left\langle (v_x - V_x)^2\right\rangle_\Gamma = \frac{k_B T_\perp(s)x_{brms}^2(s)}{m\gamma_b\beta_b^2 c^2},$$

(6)

$$\varepsilon_{yth}^2(s) \equiv (\beta_b c)^{-2}\left\langle y^2\right\rangle_\Gamma \left\langle (v_y - V_y)^2\right\rangle_\Gamma = \frac{k_B T_\perp(s)y_{brms}^2(s)}{m\gamma_b\beta_b^2 c^2},$$

(7)

in the x- and y- directions, respectively. Here, the statistical average of χ is defined in the usual manner by $\left\langle\chi\right\rangle_\Gamma = N_b^{-1}\int\chi f_b dxdydv_x dv_y$ with f_b being the equilibrium particle distribution function. The adiabatic condition in Eq. (1) implies that $\varepsilon_{xth}\varepsilon_{yth}=const$. It follows that the 4D rms thermal emittance ε_{4Dth} is a constant, i.e.,

$$\varepsilon_{4Dth}^4(s) \equiv (\beta_b c)^{-4}\left\langle x^2\right\rangle_\Gamma\left\langle y^2\right\rangle_\Gamma\left\langle (v_x - V_x)^2\right\rangle_\Gamma\left\langle (v_y - V_y)^2\right\rangle_\Gamma$$

(8)

$$= \left[\frac{k_B T_\perp(s)x_{brms}(s)y_{brms}(s)}{m\gamma_b\beta_b^2 c^2}\right]^{1/2} = const$$

By demanding that the total number of particles per unit axial length is conserved, we can determine

$$f(s) \equiv \frac{C}{x_{brms}(s)y_{brms}(s)}$$

(9)

where C is a constant of integration.

EXAMPLE OF A WARM-FLUID BEAM EQUILIBRIUM

We developed a numerical code to solve the beam density and potential profile as well as the beam envelope equations.

As an example, we consider a thermal beam focused by a periodical quadrupole magnetic focusing field with the periodic step function $\kappa_q(s) = \kappa_q(s+S)$ with a filling factor of $\eta=0.3$ and a strength of $S^2\kappa_q(0)=15$ [please see Ref. 4 for the definition of the step function].

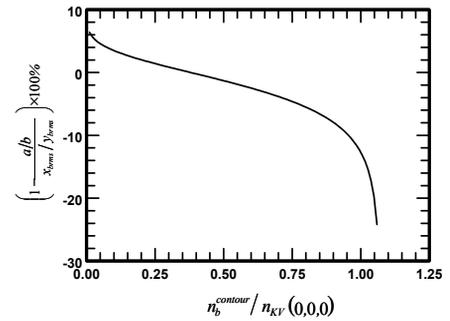We use the numerical code to calculate density profiles.

Figure 2: Plot of the difference between the ratio of the semi-axis of the contours of constant density and the ratio of the rms envelopes sizes in percent.

Although the constant-density contours are ellipses, the density profile does not satisfy the simplest elliptical symmetry condition. This is illustrated in Fig. 2 for a thermal beam with $\hat{K}=4$ at s=0, where the percentage difference between the ratio of the semi-axes of constant-density contour -a/b, and the ratio of the rms envelopes -x_{brms}/y_{brms}, is plotted as a function of the density. It can be observed in Fig. 2 that the ratio of a/b decreases as the density decreases.

In Fig. 3 we plot the beam density profiles along the x-axis and y- axis, respectively, for the same beam as shown in Fig. 2. The normalized rms beam envelopes for this beam at s=0 are $x_{brms}/\sqrt{4\varepsilon_{4Dth}S}$ =1.278 and $y_{brms}/\sqrt{4\varepsilon_{4Dth}S}$ =0.785. The beam density profile is flat near the center of the beam and then it falls off within a few Debye lengths. Here, the Debye length is defined as $\lambda_D \equiv \sqrt{\gamma_b^2 k_B T_\perp(s)/4\pi q^2 n_b(0,0,s)}$, which is evaluated to be $\lambda_D/\sqrt{4\varepsilon_{4Dth}S} = 0.171$.

It appears in Fig. 3 that the density falls off approximately at the same rate in the x- and y- directions. To quantitatively characterize the density fall-off, we compute the derivatives of the density profiles in the x- and y- directions at the points where the density is equal to half of the peak density. For the beam density profile in Fig. 3, we find that the difference between the slopes in the x- and y- directions is 8.6%. The beam density profile falls off slightly faster in the y- direction. Although the density does not have the simplest elliptical symmetry (which is a key assumption in the classic derivation of the rms envelope equations [5]), the constant-density contours are ellipses.
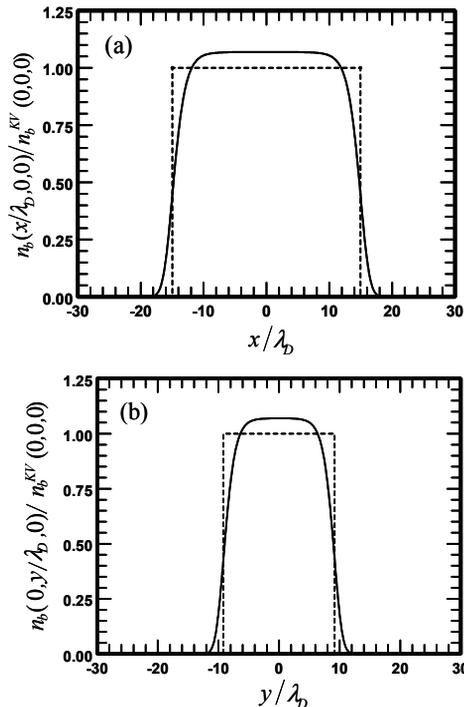

Figure 3: Plot of the beam density profile along (a) x- axis and (b) y- axis for the same beam as shown in Fig. 2. The dashed curves are the equivalent KV beam distribution.

CONCLUSIONS

We presented a warm-fluid equilibrium theory for a thermal beam in a periodic quadrupole magnetic (AG) focusing field. We considered an adiabatic process and solved the warm-fluid beam equilibrium equations in the paraxial approximation. Because the thermal beam equilibrium is adiabatic, the 4D rms thermal emittance of the beam is conserved, but the 2D rms thermal emittances are not constant. The rms beam envelope equations were derived. Although the present rms beam envelope equations have the same form as the previously known rms beam envelope equations, the evolution of the rms emittances in the present theory is given by analytical expressions. The density does not have the simplest elliptical symmetry, but the constant-density contours are ellipses. For high-intensity beams, the beam density profile is flat in the center of the beam and falls off rapidly within a few Debye lengths, and the rates of density change in the transverse directions are approximately the same.

ACKNOWLEDGMENTS

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