2D POTENTIAL FOR AN ELLIPTICAL CHARGE DISTRIBUTION

K. Haghighi Mood, Plasma Physics Research Centre, Science and Research Branch (IAU), Iran
M. Aslaninejad[#], Imperial College, London, United Kingdom and School of Particles and
Accelerators (IPM), Tehran, Iran

Abstract

2D potential profiles for uniformly populated discs of charged particles with circular and elliptical cross sections inside a perfectly conducting ring are simulated using the method of images. The results are compared with the problem of infinitely long linear charge distribution inside a conducting cylinder with a dependence only on the two transverse coordinates.

INTRODUCTION

The use of method of images to obtain the potential of a symmetric system has been well established [1]. One well cited assertion in the literature is the issue of using such a method for obtaining potential and electric field of an unbunched circular or elliptical cross section beams inside the vacuum chamber of an accelerator [2-3].

In Ref. [3], based on Cauchy's theorem, formalism for the electric field of an elliptical charge distribution confined inside a perfectly conducting circular cylinder was developed. A static charge distribution, infinitely long along the direction perpendicular to the $x-y$ plane was considered. The distribution depended only on the two coordinates x and y. These assumptions could reduce the issue to a 2D problem. It was argued that due to dependence on distance of the form $1/r$ in 2D problems for the electric field which naturally can be related to Cauchy's theorem, the electric field calculations were more preferable. This was also the case to avoid dealing with a much more complicated problem of potential which has a dependence of the form $\Phi \approx \ln r$. For a point charge at location $z' = (x' + iy')$ inside a cylinder of radius R and a central axis passing through the origin of (x, y), plane an image point charge of *the same magnitude* and opposite sign at $z'_i = \dfrac{R^2}{\bar{z}'}$ outside the cylinder was considered and it was shown that the relation between a general charge distribution $\rho(z)$ contained inside the cylinder and the image distribution $\rho(z_i)$ can be expressed as:

$$\rho_i(z_i) = -\frac{R^4}{|z_i|^4}\rho(z) \qquad (1)$$

This relation shows the inverse fourth power of the distance.

Here in this article we have attempted to address the same problem using the method of images. But, we have directly tackled the potential problem in 2D real space. We have simulated the potential profiles for different cases of a point charge, hollow rings and uniformly distributed charged particles in shapes of circular/elliptical discs in two dimensions inside a conducting circular ring with radius R.

METHOD

Due to symmetry we have associated with each point charge q at distance r from the centre of the ring an image point charge q' at the distance $r' = \dfrac{R^2}{r}$ with *the different magnitude* $q' = -q\dfrac{R}{r}$ and opposite sign located outside the ring [4]. We first consider uniformly distributed charged particles on a circular/elliptical ring inside the conductor. Note that we have here considered an intrinsic 2D problem. The potential profile can be obtained through the method of images. The corresponding image of any point on the circumference of the circle (ellipse) with respect to the ring with radius R is now obtained (see Figure 1).
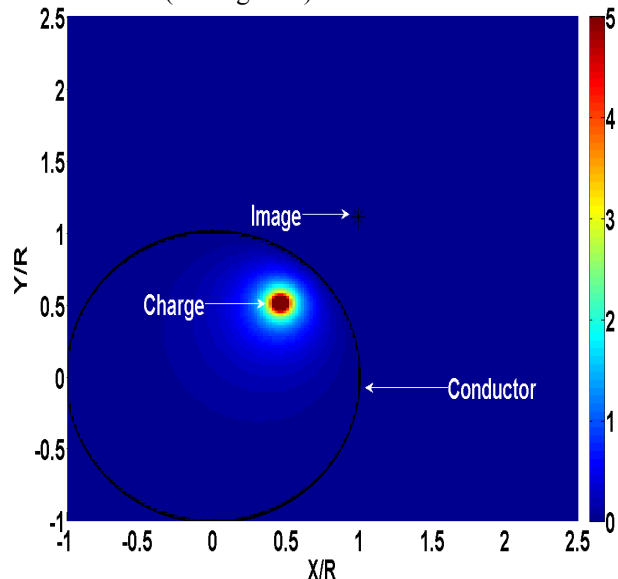

Figure 1: Potential profile for a point charge and its image in the conducting ring.

Although the relation between the positions of the charge and its image in our formalism and in Ref. [3] both read as $r' = \dfrac{R^2}{r}$, the magnitudes of the charge and its image in the latter are equal whereas in the former they

satisfy $q' = -q\dfrac{R}{r}$. As mentioned, the problem in Ref. [3] reduces to a 2D issue because the assumed charge distribution depended only on the two coordinates x and y and is infinitely long along the direction perpendicular to the $x-y$ plane. In contrast to Ref. [3] we have investigated an intrinsic 2D problem to compare the results of both formalisms. The number of charged particles on the rings is approximately 100. The potential at any point inside the ring can be obtained using the principle of superposition. The results are shown in Figure 2 and Figure 3.
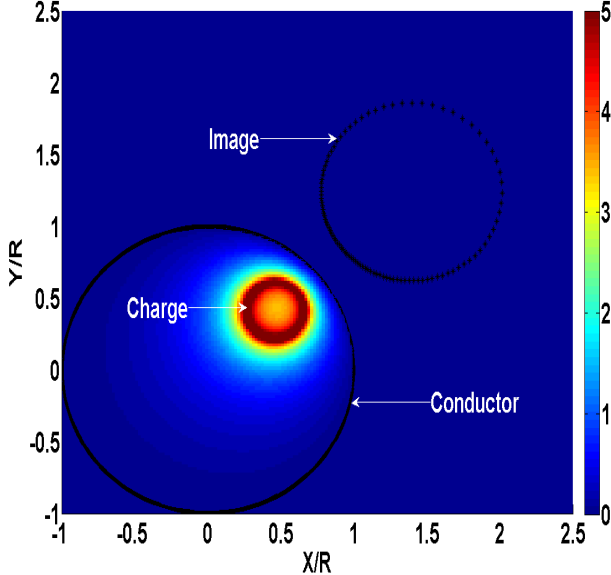

Figure 2: Potential profile for a round hollow distribution and its image with respect to the conducting ring.

Results are in complete agreement with analytical calculation of Ref. [3]. Note that as stated in Ref. [3], the image of any given circle is a circle, but the image of the centre of any given circle is not the centre of the image of the circle. This is a consequence of the inverse 4th power of the distance and becomes clearer in our next figures. Also the image of an elliptical ring is a mussel-shell shape as the curve of image is quartic and not quadratic [3]. Next we turn our attention to uniformly populated discs of charged particle with circular or elliptical cross sections inside the conducting ring. In order to homogenously spread charged particles on the discs we have used random numbers and resorted to the Neumann Rejection Method (NRM) to generate random points [5]. First a square and a rectangle around the circle and the ellipse have been considered, respectively, and were homogenously filled and then the samples satisfying:

$$(x_s - x_c)^2 + (y_s - y_c)^2 < r_d^2 \qquad (2)$$

$$\frac{(x_s - x_c)^2}{a^2} + \frac{(y_s - y_c)^2}{b^2} < 1 \qquad (3)$$

were collected. $x_s, y_s, x_{c,e}, y_{c,e}$, being the sample positions and the centres of the circular and elliptical discs, respectively. Also a, b and r_d are the semi axes and the radius for the ellipse and the circle, respectively. The total number of simulated particles is approximately 5000 of which 1000 particles were rejected. The results for the potential profiles are shown in Figures 4 and 5.

It is obvious from these results that the charge density of the image distribution is higher at the edge closer to the ring than at the farther edge which is in full agreement with the results in Ref. [3]. The potential profile shows the maximum potential is inside the charge distribution and decreases toward the edge of the conducting ring. It vanishes on the ring.
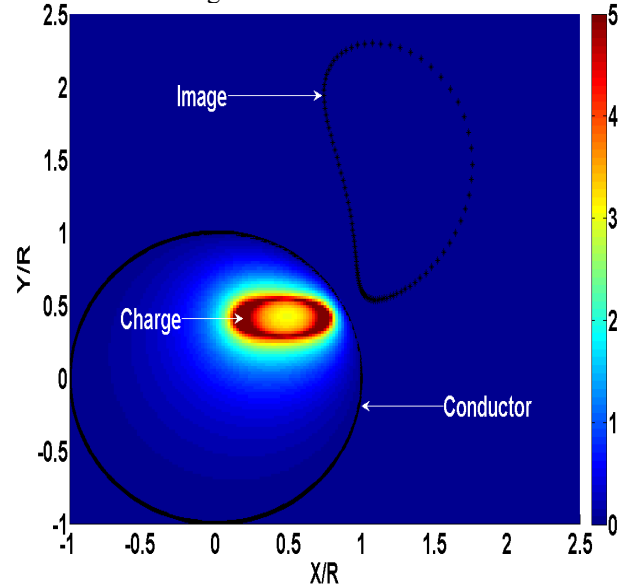

Figure 3: Potential profile for an ellipse and its mussel-shape image charge with respect to the conducting ring.
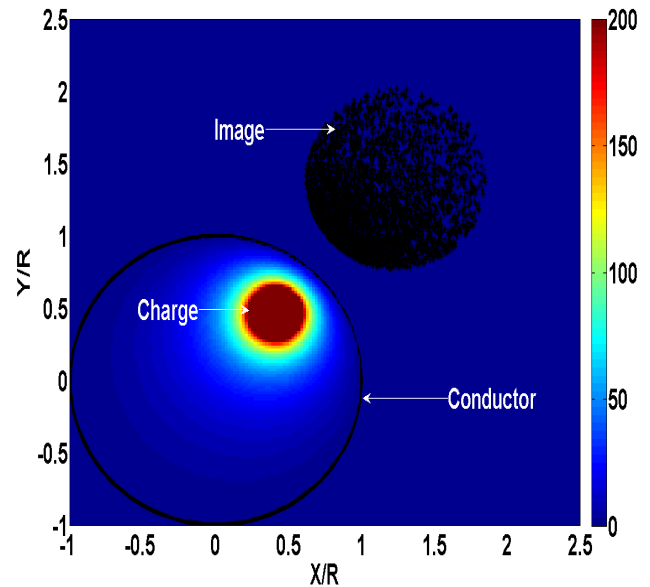

Figure 4: Potential profile for a uniform charge distribution disc and its image outside the conducting ring. The image charges are more concentrated toward the ring border.
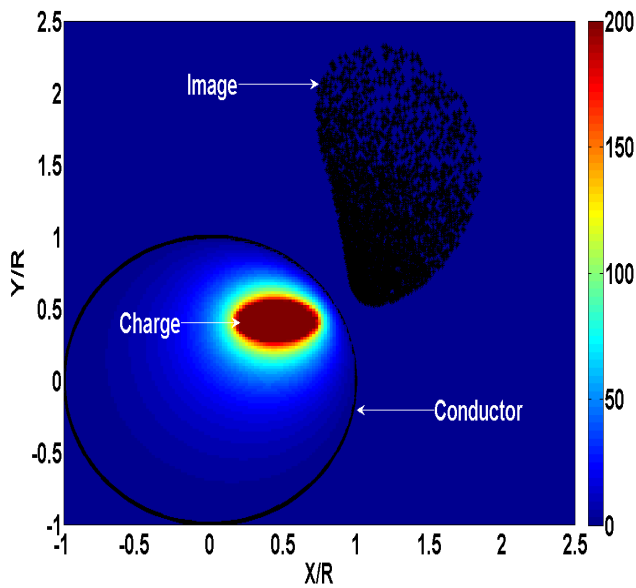
Figure 5: Potential profile for elliptical uniform charge distribution disc and its image with respect to the conducting ring. The image charges are more concentrated toward the ring border.

RESULTS AND DISCUSSION

Our 2D simulation considers a uniform charge distribution inside a conducting ring and finds the image charge and the potential profiles. Our results are similar to the analytical results of the Ref. 3. For a round ring the image is a round ring, while for an ellipse it would be a mussel shell shape. Simulation of uniform distribution of charged particle inside a round ring or an ellipse as uniform discs of particles clearly shows that the charge density of the image distribution is higher at the edge closer to the ring than at the farther edge. Potential profiles in colour show the decrease of potential from centre of the charge distribution toward the ring where it vanishes to satisfy the boundary condition. Note that our straightforward 2D approach provides the same results as those obtained when modelling a long charge distribution inside a cylinder with a transverse dependence. In the latter, which is an important problem in accelerator physics, one resorts to the electric field of an infinitely long charge distribution with only transverse dependence and equal image charge which satisfies the boundary condition. Here, we straightforwardly calculated the potential of point charges and the principle of superposition resulted in the right potential profile. The simulation for a Gaussian distribution is in progress.

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