EQUILIBRIUM TAIL DISTRIBUTION DUE TO TOUSCHEK SCATTERING

B. Nash, S. Krinsky, Brookhaven National Laboratory, Upton, NY 11973, USA

Abstract

Single large angle Coulomb scattering is referred to as Touschek scattering. In addition to causing particle loss when the scattered particles are outside the momentum aperture, the process also results in a non-Gaussian tail, which is an equilibrium between the Touschek scattering and radiation damping. Here we present an analytical calculation for this equilibrium distribution.

INTRODUCTION

Electrons in a storage ring emit radiation which results in both a damping process and a diffusion process. The effects of these processes, together with the symplectic motion through the magnets results in a Gaussian distribution which has been well studied.

Given such a distribution, however, there are other processes occuring. One is the scattering of particles off of each other. The effect of multiple small scatters results in a slow change of the beam core referred to as Intrabeam Scattering. The single larger amplitude scatters results in particle loss and is referred to as the Touschek Lifetime. In this paper, we consider the fate of those single scattered particles that are not lost due physical, or dynamic aperture.

In particular, we will consider the core beam as a source for populating the beam halo through Touschek scattering. After a scatter, the particle will damp down, returning to the core. Thus, any region of longitudinal phase space will be continually repopulated by Touschek scatters, while at the same time losing particles due to damping to smaller amplitudes and gaining particles due to damping from above. Together these effects result in an equilibrium tail distribution for a fixed core distribution.

HALO DISTRIBUTION DUE TO TOUSCHEK SCATTERING

Core Distribution

We introduce action-angle variables (J, ϕ), via

$$\delta = \sqrt{\frac{2J}{\beta_z}}\cos\phi \tag{1}$$

$$z = \sqrt{2J\beta_z}\sin\phi \tag{2}$$

Here, $\beta_z = \frac{C\alpha_c}{2\pi\nu_s}$, C is the ring circumference, α_c is the momentum compaction, and ν_s is the synchrotron tune.

We divide the electron longitudinal distribution into a core and tail distribution:

$$N(J,t) = N_C(J,t) + N_T(J,t) \tag{3}$$

Let us denote by $\bar{N}_C(t)$ and $\bar{N}_T(t)$ the total number of particles contained in each part of the distribution:

$$\bar{N}_C(t) = \int dJ N_C(J,t) \quad \bar{N}_T(t) = \int dJ N_T(J,t) \tag{4}$$

If we denote the number of particles that have been lost to physical apertures by $\bar{N}_L$, then

$$\bar{N}_C + \bar{N}_T + \bar{N}_L = N_P = \text{const} \tag{5}$$

Let us denote by $\mathring{N}(J)$ the number of particles scattering out of the core N_C into the amplitude J per unit time. Let $\hat{\delta}$ be the maximum value of δ over a synchtron oscillation. Then $J = \frac{1}{2}\beta_z\hat{\delta}^2$. Further, let $\hat{\delta}_{\rm acc}$ be the momentum acceptance (i. e. the minimum value of $\hat{\delta}$ such that the particle is lost. Now we have

$$\frac{d\bar{N}_c}{dt} = -\int_{\hat{\delta}_{\rm acc}}^{\infty} d\hat{\delta}\mathring{N}(\hat{\delta}) \equiv -\frac{\bar{N}_C^2}{T(\hat{\delta}_{\rm acc})} \tag{6}$$

where the latter results from Touschek scattering and $T(\hat{\delta})$ is a quantity independent of the number of particles in the core, with units of time. Solving this equation, and letting $N_0 = \bar{N}_C(t=0)$, we find

$$\bar{N}_C(t) = \frac{N_0}{1 + \frac{N_0 t}{T(\hat{\delta}_{\rm acc})}} \tag{7}$$

The time it takes to lose half of the core particles, $\tau_{\frac{1}{2}}$, which is the quantity standardly known as the Touschek lifetime, is given by

$$\tau_{\frac{1}{2}} = \frac{T(\hat{\delta}_{\rm acc})}{N_0} \tag{8}$$

Combining with (6) for $t = 0$, we find

$$\frac{1}{\tau_{\frac{1}{2}}(\hat{\delta}_{\rm acc})} = -\frac{1}{\bar{N}_C}\int_{\hat{\delta}_{\rm acc}}^{\infty}\mathring{N}(\lambda)d\lambda \tag{9}$$

To describe the dynamics of the tail, we assume that the core N_C is not changing much, and then use a Fokker-Planck equation with an additional source term $\mathring{N}$ which we derive later. Here, we let $N = N_T$, the number of particles in the tail. Then,

$$\frac{\partial N(z,\delta,t)}{\partial t} = \frac{\partial H}{\partial z}\frac{\partial N}{\partial \delta} - \frac{\partial H}{\partial z}\frac{\partial N}{\partial \delta} + \alpha\frac{\partial}{\partial\delta}(\delta N) + \mathring{N} \tag{10}$$

where α is the radiation damping rate and we have neglected quantum excitation since we assume that the damping effect will dominate for large amplitudes.

Assuming $N = N(J)$, and averaging over ϕ, one finds:

$$\frac{\partial N(J,t)}{dt} = \alpha\frac{\partial}{\partial J}(JN) + \mathring{N} \tag{11}$$

The solution to this equation is given by

$$N(J,t) = N_0(Je^{\alpha t})e^{\alpha t} + \int_0^\infty dx \mathring{N}(Je^{\alpha x})e^{\alpha x} \quad (12)$$

where $N_0(J) = N_0(J,0)$ is the tail distribution at $t = 0$. Consider the case where initally the tail is empty ($N_0(J) = 0$). Then

$$N(J,t) = \frac{1}{\alpha J}\int_J^{Je^{\alpha t}} \mathring{N}(\lambda)d\lambda \quad (13)$$

Noting a similarity to expression for the Touschek lifetime, (9), and using the following relationships for changing variables:

$$\mathring{N}(J)dJ = \mathring{N}_1(\hat{\delta})d\hat{\delta} \quad (14)$$
$$N(J)dJ = N_1(\hat{\delta})d\hat{\delta} \quad (15)$$

Where the 1 represents the distribution in the new variable and we drop after this. Letting $t \to \infty$ in (13), the equilibrium distribution is

$$N_{\text{eq}}(\hat{\delta}) = \frac{2}{\alpha\delta}\int_{\hat{\delta}}^\infty d\hat{\delta}'\mathring{N}(\hat{\delta}') \quad (16)$$

and applying (9), we find an equilibrium distribution

$$N_{\text{eq}}(\hat{\delta}) = \frac{2\bar{N}_C}{\alpha\hat{\delta}}\frac{1}{\tau_{\frac{1}{2}}(\hat{\delta})} \quad (17)$$

$\mathring{N}(\delta)$ *from Scattering*

We work in the beam frame. The number of particles scattering per unit time into a solid angle $d\Omega = d\cos\chi d\phi = dud\phi$ is

$$\dot{N}(\vec{x},v,v_z) = vN_0\rho(\vec{x})\frac{d\sigma}{d\Omega} \quad (18)$$

with N_0 the total number of particles and ρ the normalized spatial distribution. For the scattering cross section, we take the Moller cross section:

$$\frac{d\sigma}{d\Omega} = \frac{r_0^2c^4}{v^4}\left(\frac{4}{\sin^4(\theta)} - \frac{3}{\sin^2(\theta)}\right) \quad (19)$$

and

$$\cos\theta = \hat{x}\cdot\hat{v}' = \sin\chi\cos\phi \quad (20)$$

Integrating (19) over ϕ, we find

$$\frac{d\sigma}{du} = \frac{r_0^2\pi c^4}{v^4}\frac{2-u^2}{|u^3|} \quad (21)$$

Now, we would like to find the number of particles scattering into a particular longitudinal velocity. This is given by

$$\dot{N}(\vec{x},v,v_z) = vN_0\rho(\vec{x})\int d\Omega\frac{d\sigma}{d\Omega}\delta(\frac{v_z}{v} - u) \quad (22)$$

or

$$\dot{N}(\vec{x},v,v_z) = vN_0\rho(\vec{x})\frac{d\sigma}{du}|_{u=\frac{v_z}{v}}\Theta(1 - \frac{v_z}{v}) \quad (23)$$

with (21), this becomes

$$\dot{N}(\vec{x},v,v_z) = \frac{r_0^2\pi c^4}{vv_z^3}(2 - \left(\frac{v_z}{v}\right)^2)\Theta(1-\frac{v_z}{v}) \quad (24)$$

Next, we need the distributions of $\vec{x}$ and v. Both are Gaussian:

$$\rho(\vec{x}) = \frac{1}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_{\bar{z}}}e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}-\frac{z^2}{2\sigma_{\bar{z}}^2}} \quad (25)$$

and

$$f(v) = \frac{1}{\sqrt{2\pi}\sigma_v}e^{\frac{-v^2}{2\sigma_v^2}} \quad (26)$$

Now, we can show that

$$\int d\vec{x}\rho^2(\vec{x}) = \frac{1}{8\pi^{3/2}\sigma_x\sigma_y\sigma_{\bar{z}}} \quad (27)$$

Thus, we have

$$\dot{N}(v_z) = \frac{r_0^2c^4N_0}{8\sqrt{2}\pi\sigma_x\sigma_y\sigma_{\bar{z}}\sigma_v}\frac{1}{v_z^3}\int_{v_z}^\infty dv\frac{(2-\left(\frac{v_z}{v}\right)^2)}{v}e^{\frac{-v^2}{2\sigma_v^2}} \quad (28)$$

Now, to derive the Touschek lifetime, we need to integrate this from the maximum stable longitudinal velocity $\tilde{v}_z$ to infinity:

$$\frac{1}{\tau} = \int_{\tilde{v}_z}^\infty \dot{N}(v_z)dv_z \quad (29)$$

After changing the order of integration, we find

$$\frac{1}{\tau} = \frac{r_0^2c^4N_0}{8\sqrt{2}\pi\sigma_x\sigma_y\sigma_{\bar{z}}\sigma_v}\int_{\tilde{v}_z}^\infty dv \quad (30)$$
$$\int_{\tilde{v}_z}^v dv_z\frac{(2-\left(\frac{v_z}{v}\right)^2)}{vv_z^3}e^{\frac{-v^2}{2\sigma_v^2}} \quad (31)$$

Doing the v_z integral, we get

$$\frac{1}{\tau} = \frac{r_0^2c^4N_0}{8\sqrt{2}\pi\sigma_x\sigma_y\sigma_{\bar{z}}\sigma_v}\frac{1}{\tilde{v}_z^2} \quad (32)$$
$$\int_{\tilde{v}_z}^\infty \frac{dv}{v}\left(1-\left(\frac{\tilde{v}_z}{v}\right)^2\left[1+\ln(\frac{v}{\tilde{v}_z})\right]\right)e^{\frac{-v^2}{2\sigma_v^2}} \quad (33)$$

Now, changing variables to $w = (\frac{\tilde{v}_z}{v})$, we find

$$\frac{1}{\tau} = \frac{r_0^2c^4N_0}{8\sqrt{2}\pi\sigma_x\sigma_y\sigma_{\bar{z}}\sigma_v}\frac{1}{\tilde{v}_z^2}\int_0^1 dw\left(\frac{1}{w}-1-\frac{1}{2}\ln(\frac{1}{w})\right)e^{\frac{-\xi}{w}} \quad (34)$$

with

$$\xi = \frac{\delta_{\text{acc}}^2}{\gamma^2\sigma_{x'}^2} \quad (35)$$

We use the following relationships between beam frame quantities and lab quantities in the ultra-relativistic limit: $\tilde{v}_z = c\delta_{\text{acc}}$, $\sigma_v = \frac{1}{\sqrt{2}}c\gamma\sigma_{x'}$, $\sigma_{\bar{z}} = \gamma\sigma_z$, $\gamma d\bar{t} = dt$. Applying these and transforming to a time in the beam frame (picking up an extra $1/\gamma$), we get

$$\frac{1}{\tau} = \frac{r_0^2cN_0}{8\pi\gamma^3\sigma_x\sigma_y\sigma_z\sigma_{x'}}\frac{1}{\delta_{\text{acc}}^2}\int_0^1 dw\left(\frac{1}{w}-1-\frac{1}{2}\ln(\frac{1}{w})\right)e^{\frac{-\xi}{w}} \quad (36)$$

where ξ is defined in (35).

Time Dependent Solution

The time dependent solution is

$$N(J,t) = \frac{1}{\alpha J}\int_J^{Je^{\alpha t}} \dot{N}(\tilde{J})d\tilde{J} \tag{37}$$

Transforming (28) into the beam frame, we have

$$\mathring{N}(\hat{\delta}) = \mathcal{A}\frac{1}{\hat{\delta}^3}\int_{\hat{\delta}}^{\infty} dq \frac{(2-\left(\frac{\hat{\delta}}{q}\right)^2)}{q} e^{\frac{-q^2}{\gamma^2\sigma_{x'}^2}} \tag{38}$$

where

$$\mathcal{A} = \frac{r_0^2 c N_0}{8\pi\gamma^3\sigma_x\sigma_y\sigma_z\sigma_{x'}} \tag{39}$$

In terms of $\hat{\delta} = \sqrt(2J/\beta_z$, we have

$$N(\hat{\delta},t) = \frac{2}{\alpha\hat{\delta}}\int_{\hat{\delta}}^{\hat{\delta}e^{\alpha t}} \dot{N}(\tilde{\hat{\delta}})d\tilde{\hat{\delta}} \tag{40}$$

This is

$$N(\hat{\delta},t) = \frac{2\mathcal{A}}{\alpha\hat{\delta}}\int_{\hat{\delta}}^{\hat{\delta}e^{\alpha t}} d\tilde{\delta}\int_{\tilde{\delta}}^{\infty} dq \frac{(2-(\frac{\tilde{\hat{\delta}}}{q})^2)}{q\tilde{\delta}^3} e^{\frac{-q^2}{\gamma^2\sigma_{x'}^2}} \tag{41}$$

Changing the order of integration, we can break the 2-D integration up into two pieces:

$$N(\hat{\delta},t) = \frac{2\mathcal{A}}{\alpha\hat{\delta}}\left[\int_{\hat{\delta}}^{\hat{\delta}e^{\alpha t}} dq \int_{\hat{\delta}}^{q} d\tilde{\delta} + \int_{\hat{\delta}e^{\alpha t}}^{\infty} dq \int_{\hat{\delta}}^{\hat{\delta}e^{\alpha t}} d\tilde{\delta}\right] \times \frac{2-(\frac{\tilde{\hat{\delta}}}{q})^2}{q\tilde{\delta}^3} e^{\frac{-q^2}{\gamma^2\sigma_{x'}^2}} \tag{42}$$

Doing the q integral and making the substitution $w = (\frac{\tilde{v}_z}{v})^2$ we get

$$N(\hat{\delta},t) = \frac{2\mathcal{A}}{\alpha\hat{\delta}^3}(A_1 - A_2) \tag{43}$$

with

$$A_1 = \int_{e^{-2\alpha t}}^{1} dw(\frac{1}{w} - 1 - \frac{1}{2}\ln(\frac{1}{w}))e^{\frac{-\xi}{w}} \tag{44}$$

$$A_2 = \int_0^{e^{-2\alpha t}} dw(\frac{1}{w}(e^{-2\alpha t} - 1) + \alpha t)e^{\frac{-\xi}{w}} \tag{45}$$

which is seen to reduce to (17) for t much greater than a damping time $1/\alpha$.

Application to NSLS-II

Let us apply these results to the case of NSLS-II. Using the NSLS-II Gaussian core, we will use (43) to find the time dependent tail distribution. One can integrate our expression for the tail particle density to see the fraction of particles contained beyond a given energy.

The parameters are as follows: $\alpha = 100$ sec^{-1}, $\beta_x = 7$ m, $\epsilon_x = 5\times 10^{-10}$ m, $\beta_y = 14$, $\epsilon_y = 10$ pm, $E = 3$ GeV, $N_0 = 8\times 10^9$, $\sigma_z = 4.5$ mm. These give a value of $\mathcal{A}$ of 1.39×10^{-7} sec^{-1}. ξ, of equation (35) is 0.366 at $\hat{\delta}_{\rm acc} = 3\%$.

In Fig. 1, the distribution as a function of $\hat{\delta}$ and t is plotted. We see that after 2 damping times, the distribution has reached an equilibrium. We have normalized the distribution based on the underlying core Gaussian to give a sense of the magnitude. The integrated total number of particles beyond a given δ is given in Fig. 2. We can see that there are several picoCoulombs of charge in the halo.

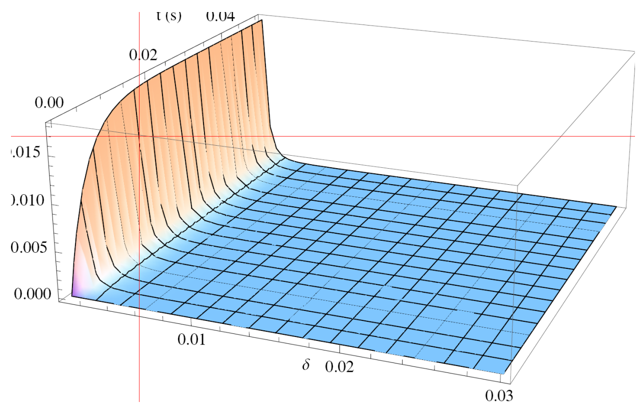

Figure 1: Approach to equilibrium of tail distribution.

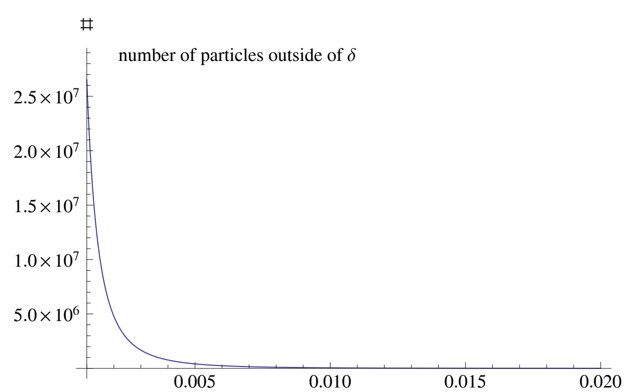

Figure 2: Total number of particles with amplitude greater than $\hat{\delta}$.

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