COMPUTATION OF RESISTIVE WAKEFIELDS FOR COLLIMATORS

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Abstract

A technique has been developed which enables the calculation of resistive particle wake effects. The technique can simply be calculated to any order, and is easy and quick to evaluate. No assumptions are made about the range of the interaction, but this is especially useful for short range effects. We show how the exact evaluation compares with various common approximations for some simple cases, and implement the technique in the Merlin and PLACET simulation programs. The extension from cylindrical to rectangular apertures is highlighted.

INTRODUCTION

Modern accelerators are increasingly demanding, with small precisely-defined beam bunches passing through small beam pipes and collimators. Thus the effects of wakefields are increasingly important and require accurate calculation to ensure they do not dilute the emittance.

The wake field is found [1] by writing down Maxwell's Equations in the aperture and in the beam pipe and matching them subject to boundary conditions. These solutions can be written as a sum over angular modes. For many purposes only the knowledge of the leading modes ($m = 0$ and $m = 1$) is adequate, however as requirements become more stringent a technique is needed for the general case.

Chao [1] gives a general formula for the impedance, from which he obtains an expression for the physical wake in the long-range limit. Bane and Sands [3, 4] extend this to shorter range though still making approximations. Various aspects have been studied by other authors [5, 6, 7]. Existing implementations use different expressions in different regimes [2]. Our approach unifies, simplifies and speeds up the calculations, and makes the underlying physics clear. It includes short range and long range wakes with no artificial division and to arbitrary angular order.

The Inverse Transform

The Fourier Transform of the $m = 0$ mode of the longitudinal component of the wakefield is given by [3]
$\tilde{E}_z(k) = \frac{2q}{b}\frac{1}{\frac{ikb}{2}-\left(\frac{\lambda}{k}+\frac{k}{\lambda}\right)\left(1+\frac{i}{2\lambda b}\right)}$,
where $\lambda(k) = \sqrt{\frac{2\pi\sigma|k|}{c}}(i + sgn(k))$, with q the charge, b the radius of the tube and σ the conductivity of the pipe. This assumes axial symmetry, the validity of Ohm's law, relativistic particles, and that the skindepth is smaller than both the thickness of the pipe and the tube radius, but is otherwise general. This enables the Bessel function solution to be replaced by a sinusoidal form. It is convenient to introduce the scaling length $s_0 = \sqrt[3]{\frac{cb^2}{2\pi\sigma}}$. and the dimensionless quantities $K = s_0 k$, and $s' = \frac{s}{s_0}$, .

To find the corresponding wakefield requires the inverse Fourier transform. This has been much studied in the literature[3, 1], using approximate forms and evaluating them by a contour integral. By contrast we will do the integration numerically, enabling us to provide a general technique without making approximations.

The back transform $E_z(s')$ can be written
$\frac{s_0}{2\pi}\int_{-\infty}^{\infty} f_{even}(K)\cos(Ks') + f_{odd}(K)\sin(Ks')\,dK$,
where the functions are the even and odd parts of $\tilde{E}_z(K)$. In the limit of large b compared to s_0 and neglecting high and low frequencies $\tilde{E}_z(k) = -\frac{2qk}{\lambda b}$. The Fourier Transform [1] is $E_z(s) = \frac{q}{2\pi b}\sqrt{\frac{c}{\sigma}}s^{-\frac{3}{2}}$. However we can also evaluate it numerically, in preparation for more complicated forms. We have $f_{even}(K) = -\sqrt{K}$, $f_{odd}(K) = i\sqrt{K}$.

A function like $\sqrt{K}\sin(K)$ presents problems for numerical integrals as the function is oscillating with increasing amplitude, and any summation technique will not work. However we know that the integral exists because it can be done analytically. We can perform the K integral by first integrating with respect to s', thereby dividing by a factor of K so that the oscillations decrease in size and the numerical integration can succeed. The result can then be differentiated numerically, using an intermediate interpolating function, to give the desired wakefield. This can be verified and results are indistinguishable from Chao's formula.

If the large s requirement is relaxed then approximately
$\tilde{E}_z(k) = \frac{2q}{b}\frac{1}{\frac{ikb}{2}-\frac{\lambda}{k}}$.
The even and odd parts are $f_{even}(K) = -\frac{\frac{2}{\sqrt{K}}}{\left(\frac{K}{2}-\frac{1}{\sqrt{K}}\right)^2+\frac{1}{K}}$, $f_{odd}(K) = -\frac{2i\left(\frac{K}{2}-\frac{1}{\sqrt{K}}\right)}{\left(\frac{K}{2}-\frac{1}{\sqrt{K}}\right)^2+\frac{1}{K}}$

Although the wake is a function of three parameters (s, b and σ), the use of the scaling length s_0 enables it to be written as a universal function. This transform can also be performed analytically, using contour integration as was done by Bane and Sands [3] and our results agree with theirs.

The Full Formula

For the full version of one gets

$$f_{even}(K) = \frac{-8\left(\xi^2+2\xi\sqrt{K}+\frac{4}{\sqrt{K}}\right)}{4\left[\xi\sqrt{K}-\frac{1}{K}\left(\xi+2\sqrt{K}\right)+K\right]^2+\left(\xi^2+\xi 2\sqrt{K}+\frac{4}{\sqrt{K}}\right)^2}$$

$$f_{odd}(K) = \frac{-16i\left[\xi\sqrt{K}-\frac{1}{K}\left(\xi+2\sqrt{K}\right)+K\right]}{4\left[\xi\sqrt{K}-\frac{1}{K}\left(\xi+2\sqrt{K}\right)+K\right]^2+\left(\xi^2+\xi 2\sqrt{K}+\frac{4}{\sqrt{K}}\right)^2}$$

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where we have introduced $\xi = s_0^2/b^2$. The earlier approximation corresponds to $\xi = 0$.

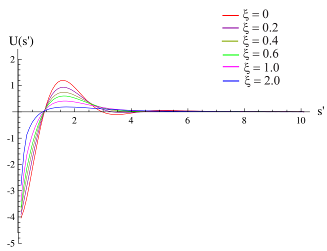

Figure 1: The exact longitudinal wake for various ξ.

Fig. 1 shows how the function when evaluated with our numerical technique changes for different values of ξ. It can be seen that for values of below about 0.1 the approximation is very good. For a copper beam pipe with a radius of 1 cm the scaling length is of order 20 microns, so ξ is very small in all practical cases at present. However, for possible future collimators with very low conductivity and small radius it might need to be considered.

Longitudinal: Higher Order Modes

The full formula for higher modes ($m > 0$) is
$\tilde{E}_z^m(k) = \frac{4I_m}{b^{2m+1}} \frac{1}{\frac{ikb}{m+1} - \left(\frac{2k}{\lambda}+\frac{\lambda}{k}\right)\left(1+\frac{i}{2\lambda b}\right) - \frac{im}{kb}}$,
This can be separated into odd and even parts and integrated numerically with our technique.

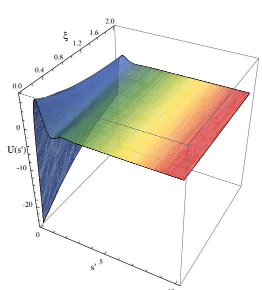

Figure 2: The $m = 5$ wake as a function of s' and ξ.

We show the dependence on ξ in Fig. 2 for $m = 5$. The dependence on ξ increases for higher modes but is still small in practical cases.

Transverse Wakes

The transverse wakefield experienced by a particle with transverse position r due to another particle at r' can be written as a sum over angular modes
$\vec{F}_T(r,\theta,s) = \sum_m r^{m-1} r'^m (\hat{r}cos(m\theta) - \hat{\theta}sin(m\theta)) W_T^m(s)$,
where θ is the angle between the two particles and s is the distance between them. The Panofsky-Wenzel theorem, applies term by term giving $W'^m_T(s) = E_z^m(s)$. The transverse wake at any order can be obtained by integrating the longitudinal wake. This is convenient as we calculate this integral for the inverse transform.

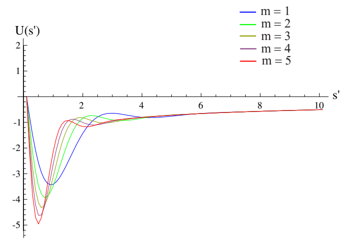

Figure 3: Transverse wakes - various modes with ξ=0.

With our formula we can study the general case. Fig. 3 shows the different transverse modes, for ξ=0.

AC Conductivity

In the classical Drude model for ac conductivity we have $\tilde{\sigma} = \frac{\sigma}{1-iK\Gamma}$, where we have introduced the dimensionless relaxation factor $\Gamma = c\tau/s_0$. For a 1 cm radius copper tube at room temperature the relaxation time is $\tau = 2.7\times10^{-14}$s or $c\tau = 8.1\ \mu$m giving $\Gamma = 0.4$, so we explore Γ values in the range 0 to 5. The dc conductivity σ is replaced by $\tilde{\sigma}$. Hence, with $t = \frac{|K|\Gamma}{\sqrt{1+K^2\Gamma^2}}$.,
$\lambda = \frac{b}{s_0^2}\sqrt{|K|}(1+K^2\Gamma^2)^{-1/4}[i\sqrt{1+t}, \pm\sqrt{1-t}]$
Fig. 4 shows the results for various values of Γ. These

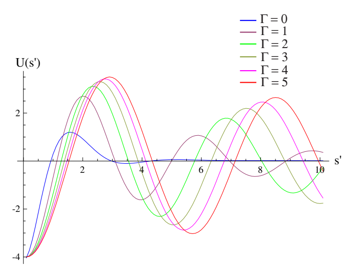

Figure 4: The longitudinal wake for various Γ values.

wakes show a fairly strong dependence of Γ and the effects of AC conductivity may be important in a particular case.

Implementation and Example

The integrals used to generate the plots in this paper were performed using Mathematica. Tables with $6 \times 1001 \times 21$ elements were written to file, covering the ranges $0 \le \Gamma \le 5$, $0 \le s' \le 100$ and $0 \le \xi \le 2$. A separate table was used for each mode, and for longitudinal and transverse wakes.

We have defined a small portable `C++` object `collimatortable`. When created it will read the full 3 dimensional table and construct a one dimensional table, using parabolic interpolation between the 9 closest points, for s' at this value of Γ and of ξ. This table can be used to find the value for any value of s' in the range. These files are obtainable from the authors.

We consider a 1 m long collimator made of Titanium, with conductivity $\sigma = 2.33 \times 10^6\,(\Omega m)^{-1}$. The radius is 1.4 mm and the relaxation time is 2.7×10^{-14} s. This gives $s_0 = 0.165\,\mu$m, $\Gamma = 0.49$ and $\xi = 0.00014$. This collimator and the beam bunch properties are those of the recent tests at SLAC End Station A, with $\beta_x = 128.2$ m, $\beta_y = 11.9$ m, $\epsilon_x = 5.49 \times 10^{-9}$ m, $\epsilon_y = 3.44 \times 10^{-10}$ m, initial beam energy $p = 28.5$ GeV and bunch length $\sigma_z = 0.3$ mm, with the sole difference of $\epsilon_x = \epsilon_y = 3.44 \times 10^{-10}$ m when using Merlin. The bunch of 10^{10} particles was modelled by 50000 macroparticles in both simulations.

We show the predictions of Merlin for the mean kick of the bunch in Fig. 5 and for PLACET for the increased geometric emittance in Fig. 6. The results from Merlin and PLACET agree surprisingly well, given that one uses the Yokoya terms and the other does not. MERLIN shows that higher modes become important (only) for offset $\geq b/2$.

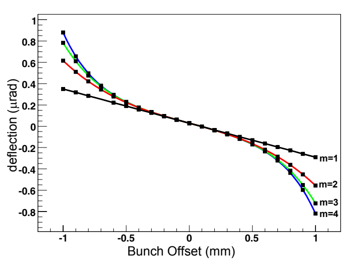

Figure 5: Merlin: deflection as a function of offset.

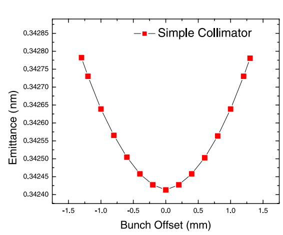

Figure 6: PLACET: emittance as a function of offset.

Rectangular Apertures

Rectangular apertures are, in Chao's words, 'considerably more complicated' owing to lack of axial symmetry. Again, one works in frequency space, and considers the fields at a witness point (x, y) due to a current of frequency ω along the z axis at a point (X, Y), for which the fields are easy to write down. It is useful to work with combinations of wires at the four points $(\pm X, \pm Y)$, distinguishing the 4 cases (++),(+-), (-+) and (- -),where the two $\pm$ signs describe the symmetry in x and y. The result of the single wire is one quarter of the sum of the 4 cases.

The fields due to the image charges satisfy the 2D Laplace equation, and have solutions of the form $sin(Kx)Sinh(Ky)$, $cosh(Kx)cos(Ky)$, etc. There are 8 possibilities but by considering each symmetry separately one restricts this to 2, for example $(++)$ must have the $cos(Kx)cosh(Ky)$ or $cosh(Kx)cos(Ky)$ form. For a slit infinite in (say) x, then the second is unphysical, however a finite rectangle requires both. Each component for E and H has this form, but the Maxwell equations mean that all 6 components can be written in terms of only 2 arbitrary functions of K, which we take as $A_x(K)$ and $A_y(K)$. We then apply the Leontovitch condition to the complete fields $\vec{E} - \hat{n}(\hat{n}.\vec{E}) = Z_s\hat{n} \times \vec{H}$. For a boundary at $y = b$ this relates E_x and H_z, likewise E_z and H_x. The two relations prescribe the two amplitudes $A_x(K)$ and $A_y(K)$. (For a rectangular slit there are boundaries at both $y = b$ and $x = a$, giving the necessary 4 conditions for the 4 amplitudes.)

This gives a wake of the form $W(k, x, y, X, Y) = \int_0^\infty G(k, K, x, y, X, Y)\,dK$. The transform from k to the longitudinal separation s can be done analytically. The integral over K requires numerical computation. Results will appear in a forthcoming publication.

Conclusions

Longitudinal and transverse resistive wakes can be calculated in simulation programs pretabulated numerical Fourier Transforms. AC conductivity and higher order angular modes are included. We have shown that this can be used by the Merlin and PLACET programs and other codes can be added in due curse. Full details of this work are given elsewhere[10] .

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