CSR IMPEDANCE DUE TO A BEND MAGNET OF FINITE LENGTH WITH A VACUUM CHAMBER OF RECTANGULAR CROSS SECTION*

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Abstract

We study the impedance due to coherent synchrotron radiation (CSR) generated by a short bunch of charged particles passing through a dipole magnet of finite length in a vacuum chamber of a given cross section. Our method represents a further development of the previous studies [5, 6]: we decompose the electromagnetic field of the beam over the eigenmodes of the toroidal chamber and derive a system of equations for the expansion coefficients in the series. We illustrate our general method by calculating the CSR impedance of a beam moving in a toroidal vacuum chamber of rectangular cross section.

INTRODUCTION

Synchrotron radiation of a relativistic beam moving in a toroidal chamber with conducting walls has been extensively studied in the past (see, e.g., [1–3]). Many important features of the radiation have been analyzed based on an exact solution of Maxwell's equations for rectangular cross section of the chamber. In a typical case when the characteristic transverse size of the vacuum chamber a is much smaller than the toroid radius R, $a \ll R$, this solution can be simplified, and the wakefield and impedance can be computed.

One of the drawbacks of the previous solutions [1–4] is that they are only applicable to a very specific geometry of the toroid, for which Maxwell's equations allow separation of variables. A different and a more general approach to the problem was proposed by the authors in Ref. [5]. It uses the smallness of the parameter $\epsilon = \sqrt{a/R}$ to simplify Maxwell's equations, keeping only terms to the lowest order in ϵ. It turns out that in this approximation the transverse components of the electric field satisfy a so called parabolic equation. In a subsequent development of the method Agoh and Yokoya added a source term to the equation due to the beam current and developed a finite difference algorithm for its numerical solution [6, 7].

In this paper we continue the analysis of Refs. [5, 6]. In contrast to numerical approach of Ref. [6], we use a mode expansions method. Throughout the paper we assume perfect conductivity of the walls, and relativistic particles with the Lorentz factor $\gamma = \infty$.

PARABOLIC EQUATION FOR THE FIELD

We consider a smooth toroidal vacuum chamber of radius R and arbitrary cross section. The geometry of the problem and the choice of the coordinate system is shown in Fig. 1a. The coordinate along the axis of the toroid is s; it is chosen in such a way that the beam propagates in the positive direction. We also use the cylindrical coordinates r and y and the notation x for the difference $x = r - R$.

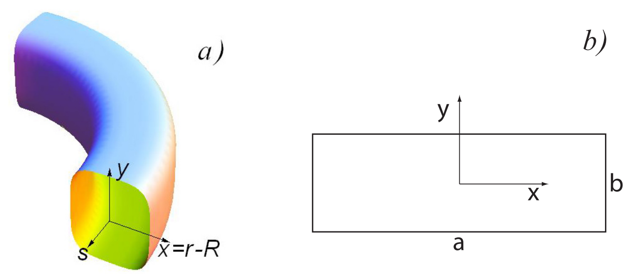

Figure 1: *a*) A section of toroidal vacuum chamber with coordinate system; *b*) cross section of a rectangular toroidal chamber with a the width in the horizontal (x) and b the height in the vertical (y) directions. The origin of the coordinate system is located at the center of the rectangle.

We will use Fourier transformed components of the field and the current defined as

$$\begin{aligned} \hat{\boldsymbol{E}}(x, y, s, \omega) &= \int_{-\infty}^{\infty} dt\, e^{i\omega t - iks}\, \boldsymbol{E}(x, y, s, t)\,, \\ \hat{j}_s(x, y, s, \omega) &= \int_{-\infty}^{\infty} dt\, e^{i\omega t - iks}\, j_s(x, y, s, t)\,, \end{aligned} \qquad (1)$$

where $k \equiv \omega/c$, and j_s is the projection of the beam current onto s. We then introduce the transverse component of the electric field $\hat{\boldsymbol{E}}_\perp$ as a two-dimensional vector $\hat{\boldsymbol{E}}_\perp = (\hat{E}_x, \hat{E}_y)$, and the longitudinal component $\hat{E}_s$.

The mathematical assumption that leads to the parabolic equation is a slow dependence of the functions $\hat{\boldsymbol{E}}_\perp$ and $\hat{j}_s$ versus s, such that $\partial/\partial s \ll k$. It means that we are interested in components of the field propagating in the positive direction of s at small angles to the axis of the toroid. In particular, we neglect a part of the field propagating in the negative direction of s. As was shown in Refs. [5, 6] this approximation is valid for calculation of wakefield at small wavelengths, and the electric field $\hat{\boldsymbol{E}}_\perp$ generated by the beam inside the toroid in this approximation satisfies the following parabolic equation

$$\frac{\partial}{\partial s}\hat{\boldsymbol{E}}_\perp = \frac{i}{2k}\left(\nabla_\perp^2 \hat{\boldsymbol{E}}_\perp + \frac{2k^2 x}{R}\hat{\boldsymbol{E}}_\perp - \frac{4\pi}{c}\nabla_\perp \hat{j}_s\right), \qquad (2)$$

where $\nabla_\perp = (\partial/\partial x, \partial/\partial y)$. The longitudinal electric field can be expressed through the transverse one and the cur-

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rent,

$$\hat{E}_s = \frac{i}{k}\left(\nabla_\perp \cdot \hat{\boldsymbol{E}}_\perp - \frac{4\pi}{c}\hat{j}_s\right). \quad (3)$$

The wall boundary conditions for Eq. (2) are imposed by the requirement that the tangential components of the electric field vanish on the metal surface of the toroid. If $\boldsymbol{n}$ is the normal vector to the surface, we have

$$\hat{\boldsymbol{E}}_\perp\Big|_w \times \boldsymbol{n} = 0, \qquad \hat{E}_s\Big|_w = 0\,. \quad (4)$$

If follows from Eq. (3) that if the density of the current on the wall vanishes, the second equation in (4) reduces to $(\operatorname{div}\hat{\boldsymbol{E}}_\perp)\big|_w = 0$ on the wall.

EIGENMODES OF TOROIDAL WAVEGUIDE AND FIELD EXPANSION

Within a framework of the parabolic equation a toroidal waveguide possesses a set of eigenmodes which are special solutions of Eq. (2) with $\hat{j}_s = 0$. Each eigenmode can be characterized by two integer indices, m and p, and the wavenumber $q_{mp}(\omega)$, which is a function of the frequency ω,

$$\begin{aligned}\hat{\boldsymbol{E}}_{mp,\perp}(x,y,s) &= \boldsymbol{\mathcal{E}}_{mp,\perp}(x,y)\,\mathrm{e}^{iq_{mp}(\omega)s},\\ \hat{E}_{mp,s}(x,y,s) &= \mathcal{E}_{mp,s}(x,y)\,\mathrm{e}^{iq_{mp}(\omega)s}\,.\end{aligned} \quad (5)$$

These modes constitute a set of orthogonal functions and can be normalized so that

$$\iint dx\,dy\left(\boldsymbol{\mathcal{E}}_{mp,\perp}\cdot\boldsymbol{\mathcal{E}}^*_{m'p'\perp}\right) = \delta_{mm'}\,\delta_{pp'}\,. \quad (6)$$

The parabolic equation is applicable if $|q_{mp}| \ll \omega/c$, which we assume throughout the paper.

In general case, for a given shape of the toroid, finding eigenmodes represents a two dimensional problem which can be solved numerically. An example of such solution for a round cross section of the toroid is given in Ref. [5]. For a toroid with rectangular cross section eigenmodes can be found analytically [5].

Following the general method of Ref. [8], we expand the perpendicular part of the electric field $\hat{\boldsymbol{E}}_\perp$ generated by the current j_s into a series

$$\hat{\boldsymbol{E}}_\perp = \sum_{p,m} C_{mp}(s)\hat{\boldsymbol{E}}_{mp,\perp}(x,y,s) \quad (7)$$

over the eigenmodes. Putting this series into Eq. (2), multiplying the result by $\hat{\boldsymbol{E}}^*_{mp,\perp}$, integrating it over the cross-section of the waveguide and using the orthogonality property of the modes leads us to an equation for the series coefficients:

$$\frac{\mathrm{d}C_{mp}}{\mathrm{d}s} = -\frac{2\pi i}{\omega}\,\mathrm{e}^{-iq_{mp}s}\iint dx\,dy\left(\nabla_\perp\hat{j}_s\cdot\boldsymbol{\mathcal{E}}^*_{mp,\perp}\right). \quad (8)$$

We will now assume that the current $\hat{j}_s$ does not vary with s and integrate (8) over s from $s = 0$ to a current value of s. The right hand side of (8) then depends on s through the exponential factor $\mathrm{e}^{-iq_{mp}s}$ which makes the integration trivial:

$$C_{mp}(s) = C_{mp}(0) + \frac{2\pi}{\omega q_{mp}}\left(\mathrm{e}^{-iq_{mp}s} - 1\right)\iint dx\,dy\left(\nabla_\perp\hat{j}_s\cdot\boldsymbol{\mathcal{E}}^*_{mp,\perp}\right)$$

Using Eq. (3), we obtain the longitudinal component of the field:

$$\begin{aligned}\hat{E}_s = \frac{i}{k}\sum_{m,p}\Bigg[&C_{mp}(0)\,\mathrm{e}^{iq_{mp}s} + \frac{2\pi}{\omega q_{mp}}(1-\mathrm{e}^{iq_{mp}s})\times\\ &\times\iint dx\,dy\left(\nabla_\perp\hat{j}_s\cdot\boldsymbol{\mathcal{E}}^*_{mp,\perp}\right)\Bigg]\nabla_\perp\cdot\boldsymbol{\mathcal{E}}_{mp,\perp} - \frac{4\pi i}{\omega}\,\hat{j}_s.\end{aligned} \quad (9)$$

BEAM ENTERING A TOROIDAL SEGMENT FROM A STRAIGHT SECTION

Equation (9) can be used to obtain a solution to the problem of a relativistic beam entering a toroidal segment from a long straight pipe of the same cross section. In the straight part, where $s < 0$, the longitudinal electric field E_s associated with the beam is equal to zero because we assume that the beam is moving with the speed of light. It is therefore equal to zero at the entrance $s = 0$ to the toroidal segment. This observation allows us to determine the initial values $C_{mp}(0)$ for the coefficients $C_{mp}(s)$.

Indeed, putting simultaneously $E_s = 0$ and $s = 0$ in Eq. (9) gives

$$0 = \frac{i}{k}\sum_{m,p} C_{mp}(0)\nabla_\perp\cdot\boldsymbol{\mathcal{E}}_{mp,\perp} - \frac{4\pi i}{\omega}\,\hat{j}_s.$$

Finding $\hat{j}_s$ from this equation and substituting it back to Eq. (9) gives

$$\begin{aligned}\hat{E}_s = \frac{i}{k}\sum_{m,p}&\left[-C_{mp}(0) + \frac{2\pi}{\omega q_{mp}}\iint dx\,dy\left(\nabla_\perp\hat{j}_s\cdot\boldsymbol{\mathcal{E}}^*_{mp,\perp}\right)\right]\times\\ &\times\left(1-\mathrm{e}^{iq_{mp}s}\right)\nabla_\perp\cdot\boldsymbol{\mathcal{E}}_{mp,\perp}.\end{aligned} \quad (10)$$

On the other hand, the perpendicular electric field of the beam at $s = 0$, which we denote $\hat{\boldsymbol{E}}_{\perp 0}$, being also continuous at this point, can be expanded into the series over eigenmodes of the toroid as $\hat{\boldsymbol{E}}_{\perp 0} = \sum_{m,p} C_{mp}(0)\,\boldsymbol{\mathcal{E}}_{mp,\perp}$. Using the orthonormality property of the modes (6), we find

$$C_{mp}(0) = \iint dx\,dy\,\left(\hat{\boldsymbol{E}}_{\perp 0}\cdot\boldsymbol{\mathcal{E}}^*_{mp,\perp}\right). \quad (11)$$

Substituting $C_{mp}(0)$ into Eq. (10) gives

$$\begin{aligned}\hat{E}_s = \frac{i}{k}\sum_{m,p}&\left[\frac{2\pi}{\omega q_{mp}}\iint dx\,dy\left(\nabla_\perp\hat{j}_s\cdot\boldsymbol{\mathcal{E}}^*_{mp,\perp}\right) - \right.\\ &\left.\iint dx\,dy\,\left(\hat{\boldsymbol{E}}_{\perp 0}\cdot\boldsymbol{\mathcal{E}}^*_{mp,\perp}\right)\right]\left(1-\mathrm{e}^{iq_{mp}s}\right)\nabla_\perp\cdot\boldsymbol{\mathcal{E}}_{mp,\perp}.\end{aligned} \quad (12)$$

If the beam travels long distance in the straight part of the pipe, $s < 0$, before entering the toroid, its electromagnetic field reaches a steady state. For an ultrarelativistic beam this field can be represented by a gradient of a scalar function: $\hat{\boldsymbol{E}}_{\perp 0} = \nabla_\perp \psi$ which obeys the Poisson equation

$$\nabla_\perp^2 \psi = \frac{4\pi}{c} \hat{j}_s. \tag{13}$$

A general solution to this equation, for rectangular cross section, can be found in Ref. [9].

RESULTS OF SIMULATIONS

We numerically computed the field of a bunch of particles with a negligible transverse size moving with the speed of light in the toroid of rectangular cross section at $x = y = 0$, see Fig. 1b. The details of the calculations will be published elsewhere [10]. The distribution function of the bunch is $Nf(s - ct)$ where N is the number of particles in the bunch; it is assumed that f is normalized by unity. This bunch is characterized by the spectrum $\hat{f}(\omega)$ $\hat{f}(\omega) = \int_{-\infty}^{\infty} dz\, \mathrm{e}^{-i\omega z/c}\, f(z)$. For a Gaussian bunch with rms length σ_z we have $\hat{f}(\omega) = \exp(-\omega^2\sigma_z^2/2c^2)$. The field of the bunch in the space-time domain is given by the inverse Fourier transform:

$$E_s(s,t) = N \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \mathrm{e}^{i\omega(s/c-t)}\, \hat{E}_s \hat{f}\,. \tag{14}$$

We express this field as a function of the variable $z = s - ct$, which is the coordinate relative to the center of the bunch at a given time, and divide it by the total charge Nq; the resulting quantity is the longitudinal wake of the bunch at location s:

$$w(z,s) = \frac{1}{q} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \mathrm{e}^{i\omega z}\, \hat{E}_s \hat{f}\,, \tag{15}$$

where $\hat{E}_s$ is given by (12). We calculated this wake on the beam orbit $x = y = 0$ only.

For the purpose of illustration, we chose the following numerical parameters: the rms bunch length of the bunch $\sigma_z = 0.5$ mm, the orbit radius $R = 1$ m, the width of the rectangle $a = 6$ cm, the height $b = 2$ cm. The head of the bunch corresponds to positive values of z.

Fig. 2 shows plots of the wakefield for four different distances s from the entrance to the toroid, close to the entrance. It demonstrates the build up of the wake when the bunch penetrates inside the magnet. Fig. 3 shows the wake for larger values of distances s. It is interesting to note that the wake in the vicinity of the location of the bunch, $|z| < 5$ mm, has reached a steady state and does not change with the distance. The wake behind the bunch, at $z \lesssim -5$ mm, did not settle down and varies with the distance s.

In conclusion, we developed a new method of calculation of the CSR wake in a toroidal vacuum chamber. The method uses the parabolic equation approximation and involves decomposition of the electromagnetic field over the modes of the toroidal waveguide. It takes into account transient effects at the entrance to the toroid from a straight pipe. As an illustrating example, we presented results of calculations of the wake for a short Gaussian bunch in a rectangular toroid.

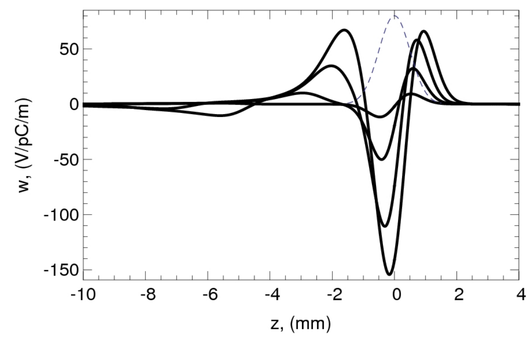

Figure 2: Wakefields for distances $s = 4, 8, 12, 16$ cm from the entrance to the toroid. The larger distances correspond to the higher amplitudes of the wake. The dashed line shows the Gaussian profile of the bunch.

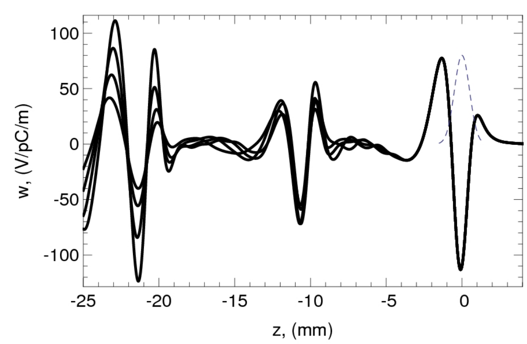

Figure 3: Wakefields for $s = 84, 88, 92, 96$ cm. The dashed line shows the Gaussian profile of the bunch.

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