# SYMPLECTIC EXPRESSION FOR CHROMATICITY

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*Abstract*

Chromaticity characterizes how optics parameters depend on momentum deviation. The optics parameters are tune, beta function and others. The chromaticity is actually defined by coefficients of the optics parameters expanding for the momentum deviation. The optics parameters including the chromaticity represent symplectic betatron motion, but are out of consideration for the synchrotron motion. We introduce Hamiltonian expression and 6 dimensional symplectic map for both of the betatron and synchrotron motion. The Hamiltonian and symplectic map are reconstructed by observable quantities, chromaticities.

## INTRODUCTION

Particles with momentum deviations experience different focusing force from that of reference particle. They also experience focusing force depending on the momentum deviations in sextupoles put in a dispersive section, because an orbit displacement is induced due to the deviations.

The linear oscillation of beam particles, betatron and synchrotron motions, are characterized by $6\times 6$ symplectic matrix, M6. The $6\times 6$ symplectic matrix contains 21 independent parameters. The matrix is parametrized with three tunes $\mu_{x,y,z}=2\pi\nu_{x,y,z}$ and 18 optics parameters. Needless to say, the tunes are independent of $s$, but optics parameters are dependent of $s$.

Momentum deviation is $\delta=(p-p_0)/p_0$ normalized by reference momentum $p_0$. As is well-known, $\delta$ varieties slowly compare to betatron variables whose form is $\mathbf{x}_\beta=(x_\beta,p_{x,\beta},y_\beta,p_{y,\beta})$, and it changes only in RF cavity. Therefore 4+1 dimensional expression, which consists of betatron motion in x-y plane and an orbit distortion due to the momentum deviation, is commonly used. The transverse coordinate of beam particle, $\mathbf{x}=(x,p_x,y,p_y)$, is expressed by

$$\mathbf{x}=\mathbf{x}_\beta+\eta\delta \tag{1}$$

where the orbit distortion is characterized by the dispersion $\eta=(\eta_x,\eta'_x,\eta_y,\eta'_y)$.

The betatron coordinates are transferred by $4\times 4$ symplectic matrix. Now the matrix is parametrized by two betatron tunes $\nu_x,\nu_y$ and 8 optics parameters $\alpha_{x,y},\beta_{x,y},r_i(i=1-4)$. The definitions of the optics parameters are as follows [1]

$$M_2=RM_{2\times 2}R^{-1} \tag{2}$$

where M2 is 4 by 4 transfer matrix which is block diagonalized by two 2 by 2 matrix with the form of Eq.(4) for x and y.

$$M_{2\times 2}=\begin{pmatrix} M_x & 0 \\ 0 & M_y \end{pmatrix} \tag{3}$$

$$M_i=\begin{pmatrix} \cos\mu_i+\alpha_i\sin\mu_i & \beta_i\sin\mu_i \\ -\gamma_i\sin\mu_i & \cos\mu_i-\alpha_i\sin\mu_i \end{pmatrix} \tag{4}$$

where $i=x,y,z$. $R$, which characterises x-y coupling, is parametrized by,

$$R=\begin{pmatrix} r_0 I_2 & -S_2R_2^tS_2 \\ -R_2 & r_0 I_2 \end{pmatrix} \tag{5}$$

$$R_2=\begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix} \qquad S_2=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{6}$$

where $r_0=\sqrt{1-\det(R_2)}$ and $I_2$ is 2 by 2 unit matrix.

Other optics parameters, 21-10-4=7, are $\nu_s,\beta_z,\alpha_z,\zeta_{x,y}$ and $\zeta'_{x,y}$. $\alpha_z\approx 0$ for accelerators with low $\nu_s<<1$. $\zeta_{x,y}$ and $\zeta'_{x,y}$, which characterize a tilt of beam, are dispersion related to $z$ (not $\delta$).

The change of the focusing force due to the momentum deviation reflected the transfer matrix and/or optics parameters. A typical example is the tune $\mu(\delta)=2\pi\nu$, whose first derivative for the momentum deviation is called the chromaticity,

$$\frac{d\nu(\delta)}{d\delta}=\xi(\delta) \tag{7}$$

The linear chromaticity is $\xi(0)$.

Needless to say, other optics parameters related to the betatron motion and x-y coupling, $\beta(\delta)$, $\alpha(\delta)$ and $r_i(\delta)$, depend on the momentum deviation. The parameters are expanded by the momentum deviation $\delta$, and their coefficients are generalized chromaticity.

The chromaticity is equivalent to an orbit lengthening due to betatron amplitude [2]: that is, the betatron motion affects the synchrotron motion via the chromaticity. Higher order chromaticity also gives momentum compaction factor depending on the momentum deviation.

In this paper, three dimensional symplectic expression for the chromaticity is discussed. The reason why the expression is worthful is summarized as follows.

Recent simulation codes are based on three dimensional formalism. The coordinates of particles are expressed by 6 variables $x,p_x,y,p_y,z,\delta$, and are tracked in the code. The expression for the chromaticity is implemented in the three dimensional codes.

In the beam-beam, space charge, electron cloud and impedance phenomena, the chromaticity sometimes important roles. The chromaticity expression, which is implemented in computer program codes for studying the phenomena, make possible to study the effects of the chromaticity. The chromaticity is observable: that is, the optics parameters can be measured in a condition with a momentum deviation. While lattice design codes like

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SAD and MAD calculate tune, twiss parameters and chromaticities. However the design values sometimes differ from measurements. The discrepancies in tune and twiss parameters are corrected by introducing fudge factors in the magnet strength for example. The correction is too complex for considering higher order chromaticity. In such case, it is better to construct to an accelerator model with the chromaticity expression.

Chromaticity causes synchro-beta resonances, which is synchrotron sideband of linear resonance [3]. Since the chromaticity expression is based on three dimensional Hamiltonian formalism, it is straightforward to discuss the synchro-beta resonance.

## LINEAR CHROMATICITY

We first discuss the simplex expression of the linear chromaticity without x-y coupling to make clear the fundamental idea. The optics parameters are $\nu,\alpha$ and $\beta$. The index $x$ or $y$ is omitted in this section.

Three chromaticities are defined by expansion $\mu,\alpha,\beta$ for $\delta$ as follows,

$$\begin{aligned}\mu &= \mu_0 + \xi_\mu\delta \equiv \mu_0 + \mu_1\delta \\ \beta &= \beta_0 + \xi_\beta\delta \equiv \beta_0 + \beta_1\delta \\ \alpha &= \alpha_0 + \xi_\alpha\delta \equiv \alpha_0 + \alpha_1\delta\end{aligned} \tag{8}$$

The transfer matrix is written by the same form as Eq.(4). This expression is symplectic for the $x$ or $y$ motion, but is not symplectic for the motion including longitudinal motion.

We consider Hamiltonian which generates the chromaticity. It is easy to be found as follows,

$$H_I(x,p_x,\delta) = \frac{ax^2 + 2bxp_x + cp_x{}^2}{2}\delta \tag{9}$$

The transformation with $H_I$ is applied after or before the linear transformation, which represent the linear betatron and synchrotron motion. It has a quadratic form of betatron coordinates multiplied by the momentum deviation. $H_I = AJ\delta$ give a pure momentum shift for the momentum deviation, but no $\beta$ or $\alpha$ shifts. $H_I = Ax^2\delta$ is regarded as a quadrupole insertion whose strength is linear for $\delta$, thus it gives distortion to not only $\mu$ but also $\beta$ and $\alpha$. The Hamiltonian contains three variables, $a,b$ and $c$, which should be related to the three chromaticities in Eqs.(8). A symplectic transformation for $H_I$ is given by using a generating function,

$$G_2(x,\bar{p}_x,\bar{\delta}) = H_I(x,\bar{p}_x,\bar{\delta}) \tag{10}$$

where $\bar{p}_x$ and $\bar{\delta}$ are $p_x$ and $\delta$ after the transformation.

The transformation using the Hamiltonian or generating function is expressed by

$$\bar{x} = x + \frac{\partial H_I}{\partial \bar{p}_x} = x + (bx + c\bar{p}_x)\bar{\delta} \tag{11}$$

$$p_x = \bar{p}_x + \frac{\partial H_I}{\partial x} = \bar{p}_x + (ax + b\bar{p}_x)\bar{\delta} \tag{12}$$

$$\bar{z} = z + \frac{\partial H_I}{\partial \bar{\delta}} = z + (ax^2 + 2bx\bar{p}_x + c\bar{p}_x^2)/2 \tag{13}$$

$$\delta = \bar{\delta} \tag{14}$$

Eq.(11) and (12) are implicit relations of $(\bar{x},\bar{p}_x)$ for $(x,p_x)$. The explicit relation is obtained by solving the two series equation as follows,

$$\begin{pmatrix}\bar{x}\\ \bar{p}_x\end{pmatrix} = M_H\begin{pmatrix}x\\ p_x\end{pmatrix} \tag{15}$$

where

$$M_H = \begin{pmatrix}1+b\delta-\dfrac{ac\delta^2}{1+b\delta} & \dfrac{c\delta}{1+b\delta}\\ -\dfrac{a\delta}{1+b\delta} & \dfrac{1}{1+b\delta}\end{pmatrix} \tag{16}$$

The transformations of Eq.(4) and/or (16) contain nonlinear terms for $\delta$ which are necessary to maintain the symplectic condition in $x$ or $y$ motion. For the symplecticity containing the longitudinal motion, Eq.(13) and the identity transformation Eq.(14) are also necessary. The set of the transformations, Eq.(16) and Eq.(13), guarantee the symplectic condition, but another set, Eq.(4) and Eq.(13) break the symplectic condition. This is the reason we introduce the Hamiltonian as Eq.(9). Eq.(13) contains $\bar{p}$, which is already determined by Eq.(16).

Here we concern a relation of $a,b,c$ and $\alpha_1,\beta_1,\nu_1$. The relation is obtained by comparison of the transfer matrices written by the chromaticities and by $M_H$, namely

$$M(\delta) = M(0)M_h(\delta) \tag{17}$$

The coefficients $a,b,c$ is expressed by

$$M_H(\delta) = M^{-1}(0)M(\delta) \tag{18}$$

Since the RHS is linear for $\delta$, $M_H$ is taken into account of linear terms for $\delta$, Now we have connection between $a,b,c$ and optics parameters, so Hamiltonian can be derived.

## XY COUPLED CHROMATICITY

In this section, we take into account of x-y coupling for the betatron oscillation. The betatron motion is represented by $4\times4$ matrix, $M_4(\delta)$. The parametrization of the x-y coupling is given in Eq.(2). Focusing force with the dependence on the energy deviation results the 4 by 4 matrix with the dependence on it,

$$M_4(\delta) = R(\delta)M(\delta)R(\delta)^{-1} \tag{19}$$

Now the coupling parameters should depend on the energy deviation, and thus they are expanded as

$$\begin{aligned}\nu(\delta) &= \sum_{n=0}\nu_n\delta^n, \beta(\delta) = \sum_{n=0}\beta_n\delta^n \\ \alpha(\delta) &= \sum_{n=0}\alpha_n\delta^n, r_i(\delta) = \sum_{n=0} r_{i,n}\delta^n\end{aligned} \tag{20}$$

where the coefficients are regarded as the chromaticity for x-y coupling.

Hamiltonian(Generating function) which gives the chromaticity is expressed by

$$\begin{aligned}H_I(x,p_x,y,p_y,\delta) = \sum_{n=1} &(a_nx^2 + 2b_nxp_x + c_np_n^2 + 2d_nxy \\ &+ 2e_nxp_y + 2f_nyp_x + 2g_np_xp_y + u_ny^2 + 2v_nyp_y + w_np_y^2)\delta^n/2\end{aligned} \tag{21}$$

$$\equiv (Ax^2 + 2Bxp_x + Cp_n^2 + 2Dxy + 2Exp_y$$
$$+2Fyp_x + 2Gp_xp_y + Uy^2 + 2Vyp_y + Wp_y^2)/2$$

The 10 set of coefficients $a_n ... w_n$ are related to $\alpha_{x,n}$, $\beta_{x,n}$, $\mu_{x,n}$, $\alpha_{y,n}$, $\beta_{y,n}$, $\mu_{y,n}$, and $r_{i,n}$, $i=1-4$. Here chromaticity related to $\alpha$, $\beta$, $\mu$ and x-y coupling are not treated independently.

The transformation due to the Hamiltonian is derived by the same way for derivation of Hamiltonian from linear chromaticity, Eq.(11)-Eq.(14),

$$\begin{aligned}
\bar{x} &= x + Bx + C\bar{p}_x + Fy + G\bar{p}_y \\
p_x &= \bar{p}_x + Ax + B\bar{p}_x + Dy + E\bar{p}_y \\
\bar{y} &= y + Vy + W\bar{p}_y + Ex + G\bar{p}_x \\
p_y &= \bar{p}_y + Uy + V\bar{p}_y + Dx + F\bar{p}_x
\end{aligned} \quad (22)$$

$$\begin{aligned}
\bar{z} = z + \sum_{n=1} (& a_n x^2 + 2b_n x\bar{p}_x + c_n\bar{p}_x^2 + 2d_n xy \\
&+ 2e_n x\bar{p}_y + 2f_n\bar{p}_x y + 2g_n\bar{p}_x\bar{p}_y + u_n y^2 \\
&+ 2v_n y\bar{p}_y + w_n\bar{p}_y^2) n\bar{\delta}^{n-1}/2
\end{aligned} \quad (23)$$

$$\delta = \bar{\delta}$$

This relation is also described by same form, Eq.(18), replaced by 4 dimensional matrix $M_H$ which is calculated by Eq.(22).

It is possible to obtain the relation between $\alpha_{x,n}$, $\beta_{x,n}$, $\mu_{x,n}$, $\alpha_{y,n}$, $\beta_{y,n}$, $\mu_{y,n}$, $r_{i,n}$ and $a_n ... w_n$ from the complex nonlinear relation. The relations for n-*th* order $a_n$ are linear for given lower order $a_{<n}$ relations. The coefficients of Hamiltonian are determined by comparison LHS coefficients with RHC coefficients, each power of $\delta$expansion, in Eq.(18).

Until now, we use Hamiltonian expression to obtain symplectic map included betatron and synchrotron motion. This form is convenient when we consider synchro-beta resonance.

## DIRECT METHOD FOR CHROMATICITY

Symplectic map is obtained from chromaticity directly without using Hamiltonian. Now we want to make symplectic map for 6 dimensional coordinates,

$$\bar{\mathbf{x}} = M_4(\delta)\mathbf{x}, \quad \bar{z} = z + g(x, p_x, y, p_y, z, \delta), \quad \bar{\delta} = \delta \quad (24)$$

where $M_4(\delta)$ (Eqs.(19)) is symplectic matrix for 4 dimensions. The symplectic map should satisfy Poison Bracket relation,

$$[\bar{x}, \bar{z}] = 0,\ [\bar{p}_x, \bar{z}] = 0,\ [\bar{y}, \bar{z}] = 0,\ [\bar{p}_y, \bar{z}] = 0 \quad (25)$$

$$[\bar{z}, \bar{\delta}] = 1 \quad (26)$$

From Eq.(26), $g$ is independent of $z$. By taking this independence into consideration, deviation of $\partial_{\mathbf{x}}\mathbf{g} = \{\partial_x g, \partial_{px} g, \partial_y g, \partial_{py} g\}$ is derived from Eqs.(25),

$$\partial_{\mathbf{x}}\mathbf{g} = -M_4(\delta)^t S(\partial_\delta M_4(\delta))\mathbf{x} \quad (27)$$

Since $M_4(\delta)$ is symplectic, following equation is satisfied,

$$\partial_\delta (M_4^t S M_4) = (\partial_\delta M_4^t) S M_4 + M_4^t S(\partial_\delta M_4) = 0 \quad (28)$$

Eq.(28) means that $M_4^t S(\partial_\delta M_4)$ is symmetric matrix. So, $g$ is derived to one form with a constant of integration which depend on only momentum deviation,

$$g = -\mathbf{x}^{\mathbf{t}} M_4(\delta)^t S(\partial_\delta M_4(\delta))\mathbf{x}/2 \quad (29)$$

## NONLINEAR DISPERSION

This formalism can be extended for nonlinear dispersion. The orbit is expanded by

$$\mathbf{x} = \mathbf{x}_\beta + \sum_{n=1} \eta_n \delta^n \quad (30)$$

Related Hamiltonian has a form as follows,

$$H_\eta = \sum_{n=1} (\eta'_{x,n} x - \eta_{x,n} p_{x,\beta} + \eta'_{y,n} y - \eta_{y,n} p_{y,\beta})\delta^n \quad (31)$$

The transformation for $H_\eta$ is expressed by

$$\begin{aligned}
x_\beta &= x + \sum_{n=1} \eta_{x,n}\delta^n, \quad p_{x,\beta} = p_x - \sum_{n=1} \eta'_{x,n}\delta^n \\
y_\beta &= y + \sum_{n=1} \eta_{y,n}\delta^n, \quad p_{y,\beta} = p_y - \sum_{n=1} \eta'_{y,n}\delta^n \\
\delta_\beta &= \delta \\
z_\beta &= \sum_{n=1} (\eta'_{x,n} x - \eta_{x,n} p_{x,\beta} + \eta'_{y,n} y - \eta_{y,n} p_{y,\beta}) n\delta^{n-1}
\end{aligned} \quad (32)$$

The last equation expresses momentum compaction which depends on betatron motion and chromaticity.

The transformation and its inverse are performed for moving from the physical coordinates to the betatron coordinates and vice versa, respectively.

## CONCLUSION

We discussed symplectic expressions for chromaticities. Symplectic map is reconstructed from observable quantities, chromaticities. The symplectic map can be useful for 6 dimensional particle tracking simulations to study the synchro-beta resonance, beam-beam, space charge, impedance effects, and so on.

In general accelerator developing codes, MAD, SAD and many others, chromaticities are calculated for ideal lattice and/or that including errors. If people include beam-beam, instability and other codes into the general developing code, the effect of the chromaticities can be evaluated basically. However measured chromaticities sometimes (or everytime) disagree with the code prediction. It is smart to perform the simulation of the beam-beam, instability and other effects with this symplectic expression based on the measured chromaticities.

## REFERENCES

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