INTEGRABLE ACCELERATOR LATTICES WITH PERIODIC AND EXPONENTIAL INVARIANTS*

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Abstract

This paper presents a new variety of one-dimensional nonlinear integrable accelerator lattices with periodic and exponential invariants in coordinates and momenta. Extension to two-dimensional transverse motion, based on a recently published approach [1], is discussed.

INTRODUCTION

The integrable accelerator lattices represent a continuation of linear systems with Courant –Snyder invariants to the nonlinear domain, where the frequencies of betatron motion "strongly" depend on betatron amplitudes (the word "strongly" means that the spread of betatron tunes is comparable to the tune itself). This spread can help to advance beam intensities by introducing a very large Landau damping. Recently, a possible method to realize stable integrable motion in accelerators with 2D transverse magnetic field was suggested [1]. In principle, all 1D integrable lattices with short nonlinear lenses can be converted to 2D integrable lattices (we'll show examples of this conversion later in this paper). Reference [2] presented a method to find a vast variety of 1D and 2D integrable systems with invariants, polynomial in coordinates and momenta. The same method was used to find invariants that are harmonic or exponential functions of coordinates and momenta. Here we briefly present the theory and the method, along with solutions for lattices having nonlinear kicks with the aforementioned invariants, and show the behaviour of these integrable lattices in the 2D case with transverse magnetic fields.

1- AND 2-CELL INTEGRABLE LATTICES

The nonlinear map we discuss here consists of linear maps and 1 or 2 thin nonlinear lenses. The linear map has a betatron phase advance of $\pi/2$ between thin nonlinear lenses and in the simplest 1-cell form is:

$$\begin{aligned} \bar{x} &= p, \\ \bar{p} &= -x + f(\bar{x}), \end{aligned} \tag{1}$$

where the bar sign denotes the new variables (papers [1,2] used somewhat different notations for that map, but it can be brought to (1) by a simple change of variables; this kind of map representation was used by E. McMillan when he discovered some integrable 1-cell maps in 1967 (see Ref. 4 in [2])). For a 2-cell lattice we have the same map (1) but the nonlinear kicks can be different – it will give us additional families of integrable lattices.

Now we present the main trick, described in [2], to get vast families of integrable systems. We start with an invariant quadratic in momentum:

$$I(x,p) = A(x)p^2 + B(x)p + C(x), \tag{2}$$

and add a nonlinear kick $\bar{p} = p + B(x)/A(x)$, where the bar sign denotes the new variables, then Eq. (2), expressed in terms of the new momentum, becomes

$$I(x,p) = A(x)\bar{p}^2 - B(x)\bar{p} + C(x). \tag{3}$$

One can see that the expression (2) is transformed into (3) in such a way that is equivalent to changing the sign of momentum. This property of the invariant was called the Sign Reversal Property (SRP) and the nonlinear kick the Sign Reversal Function (SRF) [2]. Functions with the SRP properties are all invariants with nonlinear kicks that are their SRF functions if they are also symmetric functions of coordinates and momenta. That is obvious from (1): the linear transformation in (1) interchanges the coordinates and momentum while changing the sign of the latter. Adding the kick (SRF function) changes the sign of the momentum again, so the expression acquires the same form in the new variables, i.e. it is an invariant of the map (1). Moreover, for the 2-cell map the symmetry property is not necessary: the interchanging of variables happens twice and any function with the SRF properties for two variables becomes an invariant for the map, where two kicks are the SRF functions of the invariant. As is clear from (2) and (3), any quadratic function in momentum and coordinate has SRF properties for both variables, so the most general form of quadratic invariant for 2-cell transformation is:

$$\begin{aligned} I(x,p) = ax^2p^2 + bxp^2 + cx^2p + \\ dxp + ex^2 + fp^2 + gx + hp \end{aligned}, \tag{4}$$

where *a, b, c, d, e, f, g, h* are arbitrary constants. The two kicks are:

$$f_1(f_2)(x) = -\frac{b(c)x^2 + dx + g(h)}{ax^2 + c(b)x + e(f)}. \tag{5}$$

This method was used in [3] to find non-polynomial invariants of the form $I(x,p) = \sum_i f_i(x)g_i(p)$, where f_i and g_i are arbitrary functions. Here we present the summary of that research. First, invariants of the form

$$I(x,p) = A(x)\exp(kp) + B(x)\exp(-kp) + C(x) \tag{6}$$

have the SRP with the kick

$$\bar{p} = p - \ln(B(x)/A(x))/k. \tag{7}$$

Second, if k is imaginary, and A, B are complex with $B(x) = A(x)^*$ (B is the complex conjugate of A), the invariant is a harmonic function of momentum:

*Research sponsored by Oak Ridge National Laboratory, managed by UT-Battelle, LLC, for the U. S. Department of Energy under Contract No. DE-AC05-00OR22725.

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$$I(x,p) = A(x)\exp(i\chi p) + A(x)^*\exp(-i\chi p) + C(x). \quad (8)$$

Here χ, $C(x)$ are real and the kick is taken from (7):

$$f(x) = -\ln(A^*(x)/A(x))/i\chi, \quad (9)$$

and it is also a real function of the coordinates.

Now, the invariants can each have the forms (2), (6), (8) in momentum, and the A, B, C functions in these expressions can be functions of the coordinate in the same form – exponential, harmonic, and polynomial. We present an example of how to build the invariants and the nonlinear kicks out of chosen combinations. Let's take the invariant as a quadratic polynomial in momentum and exponential function in coordinate:

$$\begin{aligned}I(x,p) &= (ap^2 + bp + c)\exp(kx) + \\ &(dp^2 + ep + f)\exp(-kx) + gp^2 + hp\end{aligned}, \quad (10)$$

where a, b... are arbitrary constants, and the invariant is taken at the beginning of the first linear transformation. Before the first kick the variables interchange and the momentum changes sign. Right before the first kick the invariant in the new coordinate and momentum is:

$$\begin{aligned}I(x,p) &= (ax^2 + bx + c)\exp(-kp) + \\ &(dx^2 + ex + f)\exp(kp) + gx^2 + hx\end{aligned}. \quad (11)$$

The momentum transformation with SRF function is:

$$\bar{p} = p - \ln((ax^2 + bx + c)/(dx^2 + ex + f))/k. \quad (12)$$

It changes the sign of the momentum. In the new variables the invariant after 1-cell is the old invariant (11) with interchanged variables. One more application of a linear map in the second cell yields invariant:

$$\begin{aligned}I(x,p) &= (ap^2 - bp + c)\exp(kx) + \\ &(dp^2 - ep + f)\exp(-kx) + gp^2 - hp\end{aligned}, \quad (13)$$

and it transforms into (10) in the new variables after the momentum transformation. The nonlinear kick (the SRF function) is

$$-\frac{b\exp(kx) + e\exp(-kx) + h}{a\exp(kx) + d\exp(-kx) + g}. \quad (14)$$

The phase space of this system with $b=a=d=c=f=1$, $e=h=g=-1$, and $k=10$ is shown in Fig. 1.

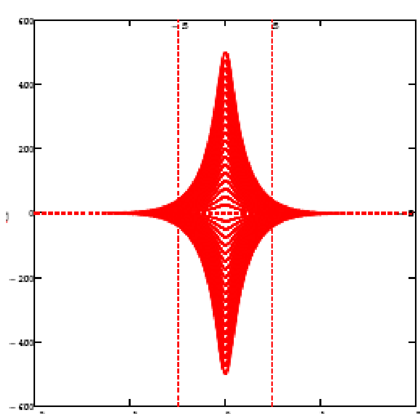

Figure 1: Phase space of the system (11).

Having built the example of mixed polynomial-exponential invariant (11), we present the general classification of all 2-cell invariants based on 3 functions:

$$\begin{aligned}&f_{1i}(t) = a_i t^2 + b_i t + c_i, \\ &f_{2j}(t) = a_j\exp(kt) + b_j\exp(-kt) + c_j, \\ &f_{3k}(t) = a_k\exp(i\chi t) + a^*{}_k\exp(-i\chi t) + c_k,\end{aligned} \quad (15)$$

where a_i, b_i, … are arbitrary constants, and $1 \le i, j, k \le 3$.

There are six types of invariants which look like this:

$$I_{kl}(x,p) = \sum_{i=1}^{3} f_{ki}(x) g_{il}(p), \quad (16)$$

where $k \le l \le 3$. The functions f_{ki} are given by Eq. (15), while the functions g_{il} are constructed from separate terms in the functions of Eq. (15):

$$\begin{aligned}&g_{11}(t) = t^2, g_{21}(t) = t, g_{31} = 1, \\ &g_{12}(t) = \exp(kt), g_{22}(t) = \exp(-kt), g_{32} = 1, \\ &g_{13}(t) = \exp(i\chi t), g_{23}(t) = \exp(-i\chi t), g_{33} = 1,\end{aligned} \quad (17)$$

and $f_{k1}(x) = f_{k2}(x)^*$ if $l=3$; in other cases all coefficients are real.

For example, the (1, 2) case is given by (10). The kicks have to be calculated as in the example of Eq. (10) – the invariant has to be transformed to the lens, and the SRF is the kick at this point, etc. Among the invariants, there are symmetric cases (we later denote them as S) for which invariant returns to itself even after the first lens. Case S(3,3) is shown in Figure 2.

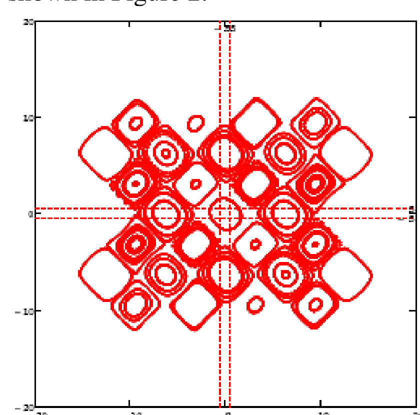

Figure 2: Phase space of the 1-cell map with periodic integrable lens.

In this case the one lens kick is:

$$f(x) = i\ln((\exp(ix) + 5)/(\exp(-ix) + 5)), \quad (18)$$

and the initial conditions cover the phase space from -10 to 10 in coordinate and momentum randomly with a step of the order of 0.1.

CONTINUATION TO 2D CASE

In [2] it is suggested how to continue all the 1D results to the 2D case, but the straightforward continuation yields unstable systems because they end up always on sum or half-sum resonances. In [1] it was proposed to make different beta-functions in x and y at the locations of the lenses. It was found by the author later that the best way to obtain 2D stable regular motion is to make the following complex map:

$$\bar{x} = p_x,\ \bar{y} = p_y / \beta,$$
$$\bar{p}_x = -x + \operatorname{Re} f(\bar{x} + i\bar{y}), \qquad (19)$$
$$\bar{p}_y = -\beta y - \operatorname{Im} f(\bar{x} + i\bar{y}),$$

where β>1 or β<1. If β=1 then the two commuting integrals are known (see [2]), but otherwise the exact expression of invariant has not yet been found. Moreover, some trajectories are found to be chaotic [4]. This means that two analytic integrals don't exist for map (19). But as was noted in [4], if one takes the 1-cell kick of the type (5), the motion in the x (if the y motion amplitude is zero) direction is integrable (and vice versa). This means that we are at least close to integrability, and it turns out that if the beta functions are separated by at least a few percent, and the closed orbit tune is far from resonance, the motion become stable and resonance-free in a large 4D volume. Figure 3 shows 4D trajectories for 3 sets of initial conditions with the kick $f(z) = -c.c.\dfrac{1.4z}{0.5z^2+1}$ in

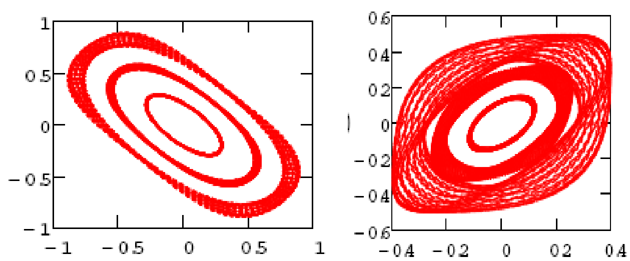

Figure 3: X (left) and Y phase spaces for 25 percent beta-function difference.

(19) and *β=1.25*.

Even though the exact invariant is not known for this case, the behaviour of the trajectories is extremely regular for not so large amplitudes. There are no signs of resonant islands or unstable motion almost up to the pole in the kick denominator. This shows that this method of continuation of 1D systems to 2D lattices with real magnetic lenses is promising. Finally, we present the sinusoidal-exponential 4D phase space. p

Figure 4 shows the case S(3,3) again for *β=1.25*, but the

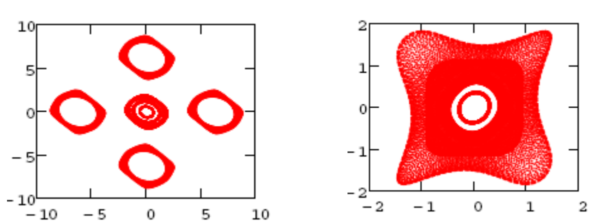

Figure 4: X (left) and Y(right) phase spaces for 25 percent beta-function difference.

kick corresponds to an exponential-sinusoidal system. In complex variables it is equal to

$$f(z) = i\ln((\exp(iz)+5)/(\exp(-iz)+5))^*.$$

One can see that in the X plane the motion resembles that of Figure 2. In the Y plane the force is exponential and it is different from the motion in the horizontal plane. Unfortunately, the space is not entirely covered by regular trajectories. Contrary to Figure 2 where separatrices are infinitely thin, the separation bands in between of regions with the regular motion have a chaotic motion. Figure 5 shows the space between eyes with regular motion. All the parameters are taken to be the same except for the

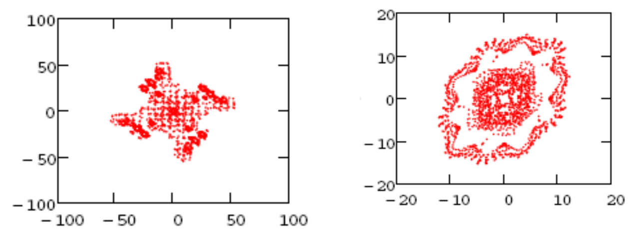

Figure 5: X (left) and Y(right) phase spaces showing a chaotic net in between of islands with regular motion.

initial conditions. One can see that the particle wanders between the regions with stable motion and its amplitude grows in a diffusion-like manner to very large values.

CONCLUSION

1-dimensional 1- and 2-cell integrable accelerator lattices with exponential and harmonic invariants are presented in this paper. It is shown how to continue these systems to 2D systems with large volumes of stable and regular motion. The problem of finding exact invariants for the stable 4D motion is not solved yet.

ACKNOWLEDGMENTS

The author is grateful to Sergei Nagaitsev for numerous useful discussions, and for initiating a quest to find exact 4D invariants of the discussed maps, and to Jeff Holmes for proofreading the manuscript.

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