NON-COMMUTATIVE COURANT-SNYDER THEORY FOR COUPLED TRANSVERSE DYNAMICS OF CHARGED PARTICLES IN ELECTROMAGNETIC FOCUSING LATTICES*

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Abstract

Courant-Snyder (CS) theory is generalized to the case of coupled transverse dynamics with two degree of freedom. The generalized theory has the same structure as the original CS theory for one degree of freedom. The four basic components of the original CS theory, i.e., the envelope equation, phase advance, transfer matrix, and the CS invariant, all have their counterparts, with remarkably similar formal expressions, in the generalized theory presented here. The unique feature of the generalized CS theory is the non-commutative nature of the theory. In the generalized theory, the envelope function is generalized into an envelope matrix, and the envelope equation becomes a matrix envelope equation with matrix operations that are not commutative. The generalized theory gives a new parameterization of the 4D symplectic transfer matrix that has the same structure as the parameterization of the 2D symplectic transfer matrix in the original CS theory.

INTRODUCTION

The transverse dynamics of a charged particle in a linear focusing lattice $\kappa_q(t)$ is described by an oscillator equation with time-dependent spring constant of the form

$$\ddot{q} + \kappa_q(t)q = 0\,, \tag{1}$$

where q represents one of the transverse coordinates, either x or y. For a quadrupole lattice, $\kappa_x(t) = -\kappa_y(t)$. The CS theory [1] gives a complete description of the solution to Eq. (1), and serves as the fundamental theory that underlies the design of modern accelerators and storage rings. There are four main components of the CS theory: the envelope equation, the phase advance, the transfer matrix, and the CS invariant. The CS theory can be summarized as follows. Because Eq. (1) is linear, its solution can be expressed as a time-dependent linear map from the initial conditions, *i.e.*, $(q,\dot{q})^\dagger = M(t)(q_0,\dot{q}_0)^\dagger$,where $q_0 = q(t=0)$ and $\dot{q}_0 = \dot{q}(t=0)$. The superscript "†" denotes the transpose operation. The transfer matrix $M(t)$ is symplectic and has the following decomposition

$$M(t) = \tag{2}$$
$$\begin{pmatrix} w & 0 \\ \dot{w} & \dfrac{1}{w} \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} w_0^{-1} & 0 \\ -\dot{w}_0 & w_0 \end{pmatrix}.$$

Here, the envelope function $w(t)$ satisfies the nonlinear envelope equation, and $\phi(t)$ is the phase advance, *i.e.*,

$$\ddot{w} + \kappa_q(t)\,w = w^{-3}\,, \tag{3}$$

$$\phi(t) = \int_0^t \frac{dt}{\beta(t)}\,,\ \beta(t) = w^2(t)\ . \tag{4}$$

The well-known CS invariant [1, 2] is

$$I = \frac{q^2}{w^2} + (w\dot{q} - \dot{w}q)^2\ . \tag{5}$$

We emphasize that the CS theory is very unique among many possible mathematical schemes to parameterize the symplectic transfer matrix. The parameters corresponding to the envelope, phase advance, and CS invariant describe the physical dimensions and the emittance of the beam, and set the foundation for many important concepts in beam physics, such as the Kapchinskij-Vladimirskij distribution function for beams with strong space-charge field.

When applying the CS theory to accelerators, the dynamics in the two transverse directions are considered to be decoupled. However, the coupling between the two transverse directions can be of considerable practical importance [3, 4]. The general form of the Hamiltonian for the coupled transverse dynamics is given by

$$H_c = \frac{1}{2} z^\dagger A_c z\,,\ z = (x, y, \dot{x}, \dot{y})^\dagger\,, \tag{6}$$

$$A_c = \begin{pmatrix} \kappa & R \\ R^\dagger & I \end{pmatrix},\ \kappa = \begin{pmatrix} \kappa_x & \kappa_{xy} \\ \kappa_{xy} & \kappa_y \end{pmatrix}. \tag{7}$$

Here, the 2×2 matrix $\kappa(t)$ is time-dependent and symmetric, R is an arbitrary, time-dependent 2×2 matrix, and I is the 2×2 unit matrix. The transverse dynamics are coupled through the $\kappa_{xy}(t)$ terms and the matrix R. A solenoidal lattice will induce non-vanishing R, and a skew quadrupole field will induce non-vanishing κ_{xy}. For a combined lattice with quadrupole, skew quadrupole, and solenoidal components, we find

$$\kappa = \begin{pmatrix} \Omega^2 + \kappa_q & \kappa_{sq} \\ \kappa_{sq} & \Omega^2 - \kappa_q \end{pmatrix},\ R = \begin{pmatrix} 0 & -\Omega \\ \Omega & 0 \end{pmatrix}, \tag{8}$$

where κ_q is the quadrupole focusing coefficient, $\Omega(t) = eB_z(t)/\gamma mc$ is the gyro-frequency associated with the solenoidal lattice, and κ_{sq} is the skew quadrupole coefficient.

The solution of the linear coupled system corresponding to H_c is given by a transfer matrix $M_c(t)$, which is a time-dependent 4×4 symplectic matrix [1]. Because there are 10 free parameters for a 4×4 symplectic matrix, many different mathematical parameterization schemes for $M_c(t)$

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exist. Teng and Edwards [5, 6, 7] first systematically studied the transfer matrix and derived various parameterization schemes [5], among which the "symplectic rotation form" [6] has been adopted in lattice design and particle tracking codes, such as the MAD code [8]. Other possible parameterizations have also been considered [9]. However, these parameterizations lack connections with the physics of the beam dynamics. They do not provide us with useful physical insights regarding the coupled dynamics. For example, these parameterization schemes do not give us effective tools to investigate the stability properties of the coupled dynamics. They also do not describe the beam envelopes for the coupled transverse dynamics, which are obviously key physical parameters of the beams. Ripken [10, 11] developed a method to describe beam envelopes for coupled dynamics without using these parameterization schemes, which attests to the ineffectiveness of the existing parameterization schemes.

In this paper, we develop a new physical parameterization of the transfer matrix $M_c\left(t\right)$ for coupled transverse dynamics by extending the CS theory for one degree of freedom to the case of coupled transverse dynamics described by the Hamiltonian H_c in Eq. (6). The generalized CS theory has the same structure as the original CS theory for one degree of freedom. The four basic components of the original CS theory that have physical importance, *i.e.*, the envelope equation, phase advance, transfer matrix, and the CS invariant, all have their counterparts, with remarkably similar expressions, in the generalized CS theory developed here. The unique feature of the generalized CS theory presented here is the non-commutative nature of the theory. In the generalized theory, the envelope function w is generalized to an envelope matrix, and the envelope equation becomes a matrix envelope equation with matrix operations that are not commutative. The generalized theory gives a parameterization of the 4D symplectic transfer matrix M_c [Eqs. (16)] that has the same structure as the parameterization of the 2D symplectic transfer matrix M [Eq. (2)] in the original CS theory.

NON-COMMUTATIVE CS THEORY

We use a time-dependent canonical transformation, first proposed by Leach [12], to develop the generalized CS theory. We consider a linear, time-dependent Hamiltonian system with n-degree of freedom given by $H = z^{\dagger}A\left(t\right)z\,/2$and $z = \left(x_1, x_2, ..., x_n, \dot{x}_1, \dot{x}_2, ..., \dot{x}_n\right)^{\dagger}$. Here, $A\left(t\right)$ is a $2n\times 2n$ time-dependent, symmetric matrix. The Hamiltonian in Eq. (6) has this form with $n = 2$. We introduce a time-dependent linear canonical transformation $\bar{z} = S\left(t\right)z$, such that in the new coordinate $\bar{z}$, the transformed Hamiltonian has the form $\bar{H} = \bar{z}^{\dagger}\bar{A}\left(t\right)\bar{z}/2$, where $\bar{A}\left(t\right)$ is a targeted symmetric matrix. Because $\bar{z} = S\left(t\right)z$ is required to be canonical, the transformation matrix is symplectic, *i.e.*, $SJS^{\dagger} = J$. In addition, the transformation $S\left(t\right)$ that renders this canonical transformation needs to satisfy [12, 13]

$$\dot{S} = \left(J\bar{A}S - SJA\right), \tag{9}$$

where J is the $2n\times 2n$ unit symplectic matrix of order $2n$. Equation (9) will play an important role in determining the structure of the generalized theory.

We are now ready to develop the generalized CS theory for coupled transverse dynamics described by the Hamiltonian H_c, using this technique of time-dependent canonical transformation. For simplicity of presentation, we only consider the case of $\Omega = 0$ in this paper. Treatments for more general cases can be found in Ref. [13]. Our objective is to solve the coupled system by determining the transfer matrix between the initial condition $z_0 = \left(x_0, y_0, \dot{x}_0, \dot{y}_0\right)^{\dagger}$ and $z = \left(x, y, \dot{x}, \dot{y}\right)^{\dagger}$ at time t. We accomplish this goal by two time-dependent canonical transformations. The first step is to transform H_c into

$$\bar{H}_c = \frac{1}{2}\bar{z}^{\dagger}\bar{A}_c\bar{z}\,,\;\; \bar{A}_c = \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta^{-1} \end{pmatrix}, \tag{10}$$

and the second step is to transform $\bar{H}_c$ into $\bar{\bar{H}}_c = 0$. Here, β is a time-dependent 2×2 matrix to be determined. As implied by its notation, the matrix β is the generalized β function for the coupled dynamics. The physics that appears in the first step is the envelope matrix and the non-Abelian matrix envelope equation. The physics that appears in the second step is the phase advance. Let $\bar{z} = Sz$ be the transformation that transforms H_c into $\bar{H}_c$. From Eq. (9), the differential equation for S is

$$\dot{S} = \left(J\bar{A}_cS - SJA_c\right). \tag{11}$$

The solution of Eq. (11) is

$$S = \begin{pmatrix} \left(w^{-1}\right)^{\dagger} & 0 \\ -\dot{w} & w \end{pmatrix},$$

where $\beta^{-1} = \left(w^{-1}\right)^{\dagger}w^{-1}$ and w is the 2×2 envelope matrix satisfying the envelope matrix equation

$$\ddot{w} + w\kappa = \left(w^{-1}\right)^{\dagger}w^{-1}\left(w^{-1}\right)^{\dagger}. \tag{12}$$

The inverse transformation is

$$z = S^{-1}\bar{z}\,,\; S^{-1} = \begin{pmatrix} w^{\dagger} & 0 \\ w^{-1}\dot{w}w^{\dagger} & w^{-1} \end{pmatrix}. \tag{13}$$

The matrix S^{-1} is the non-commutative generalization of the first matrix in the expression of the transfer matrix M for the original CS theory, *i.e.*, the first term on the right-hand side of Eq. (2).

The next step is to transform $\bar{H}_c$ into $\bar{\bar{H}}_c = 0$ with $\bar{\bar{A}}_c = 0$ by a transformation specified by $\bar{\bar{z}} = P\bar{z}$. Following the same procedure, the differential equation for P is

$$\dot{P} = P\dot{\phi}\,,\; \dot{\phi} \equiv \begin{pmatrix} 0 & -\left(w^{-1}\right)^{\dagger}w^{-1} \\ \left(w^{-1}\right)^{\dagger}w^{-1} & 0 \end{pmatrix}, \tag{14}$$

which admits solution of the form $P = \left(\begin{smallmatrix} P_1 & P_2 \\ -P_2 & P_1 \end{smallmatrix}\right)$. From the fact that P belongs to $Sp\,(4, R)$, we can readily show that $PP^\dagger = I$, and $Det\,(P) = 1$. Therefore, P corresponds to a rotation in the 4D phase space, $P \in SO\,(4)$. In this sense, $P^\dagger$ is the 4D non-commutative generalization of the 2D rotation matrix in the expression for the transfer matrix M in the original CS theory, *i.e.*, the second term on the right-hand side of Eq. (2). Because $\dot{\phi}^\dagger = -\dot{\phi}$, it follows that $\dot{\phi}$ belongs to the Lie algebra $so\,(4)$, *i.e.*, $\dot{\phi}$ is an infinitesimal generator of a 4D rotation. In another word, $\dot{\phi}$ is an "angular velocity" in 4D space, which is equivalent to a phase advance rate in 4D space. The 4D phase advance rate $\dot{\phi}$ is determined from the 2×2 matrix $\beta^{-1} = \left(w^{-1}\right)^\dagger w^{-1}$, which is remarkably similar to the the phase advance rate $\beta^{-1} = 1/w^2$ in the original CS theory for one degree of freedom [see Eq. (4)].

Because $\bar{\bar{H}}_c = 0$, the dynamics of $\bar{\bar{z}}$ is trivial, *i.e.*, $\bar{\bar{z}} = \bar{\bar{z}}_0$, and we have solved the Hamiltonian system H_c in $\bar{\bar{z}}$. From $\bar{\bar{z}} = PSz$ and $\bar{\bar{z}} = \bar{\bar{z}}_0$, we obtain the linear map between z_0 and z, *i.e.*,

$$z = S^{-1}P^{-1}\bar{\bar{z}} = S^{-1}P^{-1}\bar{\bar{z}}_0 = S^{-1}P^{-1}P_0S_0z_0\,. \quad (15)$$

Because $P \in SO(4, R)$, without loss of generality we select the initial condition $P_0 = P\,(t = 0) = I$, to obtain $z = M_c z_0$,

$$M_c = S^{-1}P^{-1}S_0 = \begin{pmatrix} w^\dagger & 0 \\ w^{-1}\dot{w}w^\dagger & w^{-1} \end{pmatrix} \begin{pmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{pmatrix} \begin{pmatrix} w_0^{-1\dagger} & 0 \\ -\dot{w}_0 & w_0 \end{pmatrix}. \quad (16)$$

The transfer matrix M_c in Eq. (16) is the 4D non-commutative generalization of the transfer matrix in Eq. (2) for one degree of freedom. The similarities between M_c and M is evident from Eqs. (16) and (2). We note that Eq. (16) has the general format of Eq. (38) of Ref. [14] using the normal form methods [15, 16, 17], which is valid for any general linear or nonlinear lattice. The specific expressions of the matrix elements in Eq. (16) are of course not valid for an arbitrary linear or nonlinear lattice. They are only correct for the coupled linear lattice under investigation here. The generalized CS invariant for 4D coupled dynamics corresponding to the original CS invariant is

$$\begin{aligned} I_{CS} &= \bar{\bar{z}}^\dagger\bar{\bar{z}} = z^\dagger S^\dagger P^\dagger PSz = z^\dagger S^\dagger Sz \\ &= \left(z^\dagger w^{-1}w^{-1\dagger}z\right) + \left(\dot{z}^\dagger w^\dagger - z^\dagger\dot{w}^\dagger\right)\left(w\dot{z} - \dot{w}z\right), \end{aligned} \quad (17)$$

where the phase advance has been removed due to the fact that P is a 4D rotation.

We now show that the generalized CS theory developed for coupled transverse dynamics recovers the original CS theory for dynamics with one degree of freedom as an special case. For the uncoupled transverse dynamics given by H_c with $\kappa_{xy} = 0$, κ is diagonal, and the matrix envelope equation Eq. (12) admits solutions with diagonal envelope matrix $w = \left(\begin{smallmatrix} w_x & 0 \\ 0 & w_y \end{smallmatrix}\right)$. Consequently, every matrix in Eq. (12) is diagonal, and the matrix operation is commutative. The matrix envelope equation reduces to two decoupled envelope equations of the conventional form for w_x and w_y, *i.e.*, $\ddot{w}_x + w_x\kappa_x = w_x^{-3}$ and $\ddot{w}_y + w_y\kappa_y = w_y^{-3}$. The 2×2 matrix of phase advance rate β^{-1} reduces to a diagonal matrix as well, *i.e.*, $\beta^{-1} = \left(\begin{smallmatrix} w_x^{-2} & 0 \\ 0 & w_y^{-2} \end{smallmatrix}\right)$. The phase advance is $P = \left(\begin{smallmatrix} P_1 & P_2 \\ -P_2 & P_1 \end{smallmatrix}\right)$, $P_1 = \left(\begin{smallmatrix} \cos\phi_x & 0 \\ 0 & \cos\phi_y \end{smallmatrix}\right)$ and $P_2 = \left(\begin{smallmatrix} \sin\phi_x & 0 \\ 0 & \sin\phi_y \end{smallmatrix}\right)$, where $\dot{\phi}_x = w_x^{-2}$ and $\dot{\phi}_y = w_y^{-2}$ are the phase advances in the x- and y-directions. Evidently, the $(x, \dot{x})$ dynamics and the $(y, \dot{y})$ dynamics are decoupled, and the transfer matrices for $(x, \dot{x})$ and $(y, \dot{y})$ extracted from the transfer matrix M are identical to that in Eq. (2) for one degree of freedom.

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