NUMERICAL SOLUTION FOR THE POTENTIAL AND DENSITY PROFILE OF A THERMAL EQUILIBRIUM SHEET BEAM *

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Abstract

In a recent paper, S. M. Lund, A. Friedman, and G. Bazouin, *Sheet beam model for intense space-charge: with application to Debye screening and the distribution of particle oscillation frequencies in a thermal equilibrium beam*, in press, Phys. Rev. Special Topics – Accel. and Beams (2011), a 1D sheet beam model was extensively analyzed. In this complementary paper, we present details of a numerical procedure developed to construct the self-consistent electrostatic potential and density profile of a thermal equilibrium sheet beam distribution. This procedure effectively circumvents pathologies which can prevent use of standard numerical integration techniques when space-charge intensity is high. The procedure employs transformations and is straightforward to implement with standard numerical methods and produces accurate solutions which can be applied to thermal equilibria with arbitrarily strong space-charge intensity up to the applied focusing limit.

INTRODUCTION

In Ref. [1], a 1D sheet beam model is extensively developed to provide a simplified framework for analysis of space-charge effects in beams. Topics covered include the (analytical) solution for the electrostatic self-field including careful identification of image charge effects arising from any conducting boundaries of the confinement geometry. A Vlasov-Poisson model is presented along with conservation constraints supported by the model and equilibrium and stability constraints are reviewed. General centroid and envelope moment equations are derived and then simplified for the case of a uniform density, rms equivalent beam and the simplified forms are compared to results from more familiar, higher dimensional models. The self-consistent 1D KV distribution which generates the uniform density beam is reviewed including extensions to include image charge effects. Rms equivalency is exploited to develop parametric equivalences to higher dimensional beam models. This general formulation is then applied to analyze a continuously focused thermal equilibrium sheet beam. Equilibrium properties are parametrically calculated and are shown to be remarkably similar to higher dimensional models in spite of the coulomb force being very different in the 1D sheet beam model relative to 2D and 3D models. The screened interaction of a test charge inserted into the thermal equilibrium distribution is shown to be essentially the same as obtained for higher dimensional beam models to help explain this equivalence. The Debye screening equivalence suggests that collective space-charge effects can be accurately represented in the 1D sheet beam model. Finally, motivated by the simple 1D space-charge model being capable of representing realistic space-charge effects, the self-consistent distribution of particle oscillation frequencies in a thermal equilibrium sheet beam is calculated and shown to become very broad for high relative space-charge intensity. This result helps explain why smooth beam distributions are observed in experiment and simulations[3, 4, 5] to have a surprising degree of stability when space-charge intensity is high.

The analysis presented in Ref. [1] is extensive and was reviewed at the 2011 PAC meeting. It makes little sense to present an abbreviated summary of parts of it in a short conference paper. Motivated by this, here we present details of a numerical procedure employed in Ref. [1] to accurately and efficiently calculate the potential and density of a thermal equilibrium sheet beam. This procedure was only briefly outlined in Ref. [1]. However, the details are useful because it is a common problem for high space-charge intensity that the potential and density of a continuously focused equilibrium distributions $f(H)$ which are a smooth function of the Hamiltonian H lead to extremely flat density profiles out to near the radial edge of the beam due to strong Debye screening of the linear applied focusing force. For high space-charge intensity, this can make direct numerical solution of the potential profile and the related density profile surprisingly problematic due to limited numerical precision and the stiff, highly-nonlinear form of the equilibrium Poisson equation. Analogous problems also occurs in higher dimensional versions of thermal and other smooth equilibrium beam distributions. Past publications typically avoided this issue by only analyzing parameters where standard numerical methods allow straightforward integration for the equilibrium potential. Unfortunately, space-charge dominated beams can typically be well within a parameter ranges where such direct integrations fail due to these numerical issues. Recently, two methods were developed to deal with this difficulty and thereby allow analysis of relevant parameter regimes to space-charge dominated beams. In Ref. [2] a method was developed for a 2D thermal equilibrium beam by using a combination of a nonlinear power series solution near the beam-axis out to a radius near the beam edge where the series is still convergent and the solution could then be numerically continued into

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the beam edge using standard integration techniques. Alternatively, in Ref. [6] an accurate, but approximate analytical solution was derived that is valid for high space-charge intensity. Both of these methods are complicated and the second is only valid for an imprecisely defined range of parameters. Here, we present an alternative numerical procedure similar, but simpler to the one applied in Ref. [2] within the context of a 1D sheet beam thermal equilibrium. This method is accurate and easy to apply and with minor modifications is applicable to a wide range of smooth distribution functions.

THERMAL EQUILIBRIUM POISSON EQUATION

Analysis of a rest frame transformation shows that the 1D Maxwell-Boltzmann equilibrium distribution $f(x, x') = f(H)$ is[1]

$$f(H) = \left(\frac{m\gamma_b\beta_b^2c^2}{2\pi T}\right)^{1/2} \hat{n}\exp\left(\frac{-m\gamma_b\beta_b^2c^2 H}{T}\right). \quad (1)$$

Here,

$$H = \frac{1}{2}x'^2 + \frac{1}{2}k_{\beta 0}^2 x^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2} \quad (2)$$

is the Hamiltonian, q and m are the ion charge and mass, x and x' are the particle coordinate and angle, c is the speed of light in *vacuo*, β_b = const and $\gamma_b = 1/\sqrt{1-\beta_b^2}$ are the axial relativistic factors of the beam, $\hat{n}$ = const is a density scale, and T = const is the thermal temperature (energy units, lab frame measure). The potential ϕ satisfies the Poisson equation

$$\frac{\partial^2}{\partial x}^2 \phi = -\frac{q}{\epsilon_0}n \quad (3)$$

where ϵ_0 is the permittivity of free-space, and

$$n \equiv \int_{-\infty}^{\infty} dx'\, f \quad (4)$$

is the beam number density. Without loss of generality we assume a potential reference $\phi(x = 0) = 0$ which identifies $\hat{n}$ with the on-axis density, i.e., $n(x = 0) = \hat{n}$. As expected, the thermal distribution (1) is consistent with a spatially uniform kinetic temperature T_x with

$$T_x \equiv m\gamma_b\beta_b^2c^2 \frac{\int_{-\infty}^{\infty} dx'\, x'^2 f}{\int_{-\infty}^{\infty} dx'\, f} = T = \text{const.} \quad (5)$$

To fully specify the equilibrium distribution $f(H)$ in Eq. (1), The potential ϕ must be self-consistently calculated from the Poisson equation (3) using the density (4) for the thermal distribution. It is straightforward to show that the resulting Poisson equation can be expressed as[1]

$$\frac{\partial^2}{\partial\rho^2}\psi = 1 + \Delta - e^{-\psi}. \quad (6)$$

Here,

$$\psi \equiv \frac{m\gamma_b\beta_b^2c^2}{T}\left(\frac{1}{2}k_{\beta 0}^2x^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}\right) \quad (7)$$

is a dimensionless streamfunction, $\rho \equiv x/(\gamma_b\lambda_D)$ is a scaled x-coordinate measured in relativistic Debye lengths with $\lambda_D = [\epsilon_0 T/(q^2\hat{n})]^{1/2}$ denoting the Debye length formed by the on-axis beam density $\hat{n}$, and

$$\Delta \equiv \frac{\gamma_b^3\beta_b^2c^2k_{\beta 0}^2}{\hat{\omega}_p^2} - 1 \quad (8)$$

is a positive dimensionless parameter ($0 < \Delta < \infty$) relating the ratio of applied to space-charge defocusing forces. In the definition for Δ in Eq. (8), $\hat{\omega}_p \equiv [q^2\hat{n}/(\epsilon_0 m)]^{1/2}$ denotes the plasma frequency formed from the on-axis beam density $\hat{n}$. The scaled Poisson equation (6) is solved for ψ subject to the "initial" conditions $\psi(\rho = 0)$ and $\partial\psi/\partial\rho|_{\rho=0} = 0$ to effectively specify the effective equilibrium potential via ψ or the scaled equilibrium density

$$\mathcal{N} = \frac{n}{\hat{n}} = e^{-\psi} \quad (9)$$

as a function of $\rho = x/(\gamma_b\lambda_D)$ in terms of the single dimensionless parameter Δ. Ultimately, the equilibrium should be specified for most applications in terms of usual accelerator parameters such as perveances and emittances. Nonlinear equations of constraint to apply such parameters can be found in Ref. [1].

Before specifying the numerical method employed to generate the solution to Eq. (6) in the next section, we present the solution in Fig. 1 for a wide range of Δ in terms of the scaled density $\mathcal{N} = \exp(-\psi)$ as a function of $\rho = x/(\gamma_b\lambda_D)$. The streamfunction ψ is easily obtained from $\psi = -\ln\mathcal{N}$ and the solution is even in ρ with $\mathcal{N}(\rho) = \mathcal{N}(-\rho)$. For very small values of $\Delta, \mathcal{N}$ is exceedingly flat ρ well away from the beam edge before dropping to exponentially small values as ρ increases by a few units (regardless of the small value of Δ) near the beam edge. How wide the flat core becomes in ρ sensitively depends on the (small) value of $\Delta > 0$. Although some values of Δ chosen in Fig. (6) appear very small, real space charge dominated beams can have exceedingly small Δ. This is illustrated in Fig. 2 where results presented in Ref. [1] are applied to plot Δ versus rms equivalent beam tune depression σ/σ_0. $\sigma/\sigma_0 \to 1$ corresponds to a warm beam with negligible space-charge strength, whereas $\sigma/\sigma_0 \to 0$ corresponds to a cold beam with space-charge nearly fully canceling the applied focusing in the core (space-charge limit) due to strong Debye screening. Evidently, $\sigma/\sigma_0 << 1$ corresponds to exceedingly small values of Δ and a very flat beam core many Debye lengths in width. In the core, $\psi \simeq 0$ before rapidly growing near the beam edge where the density abruptly drops. This highly nonlinear structure can demand extreme precision and result in problems when numerically integrating Eq. (6) for ψ as a function of ρ from the initial conditions $\psi(\rho = 0) = 0 = \partial\psi/\partial\rho|_{\rho=0}$.

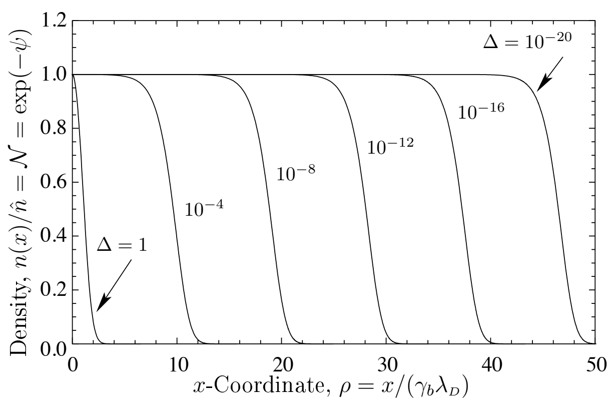

Figure 1: Solution for scaled density $\mathcal{N} = n/\hat{n}$ versus scaled radial coordinate $\rho = x/(\gamma_b\lambda_D)$ for indicated values of the dimensionless space-charge parameter Δ [1].

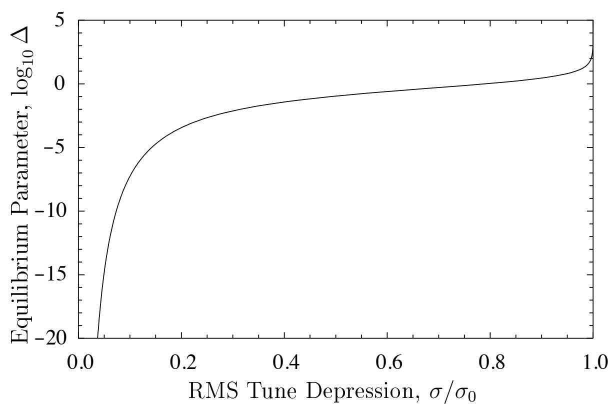

Figure 2: Dimensionless space-charge parameter Δ versus rms equivalent beam tune depression σ/σ_0 [1].

NUMERICAL SOLUTION FOR THE DENSITY AND POTENTIAL

To formulate a method to circumvent the issues outlined with direct numerical integration of Eq. (6) for ψ for small $\Delta \ll 1$, we first express Eq. (6) in terms of $\mathcal{N} = exp(-\psi)$ which leads to

$$\frac{1}{\mathcal{N}}\frac{\partial^2\mathcal{N}}{\partial\rho^2} - \frac{1}{\mathcal{N}^2}\left(\frac{\partial\mathcal{N}}{\partial\rho}\right)^2 = \mathcal{N} - 1 - \Delta, \qquad (10)$$

subject to $\mathcal{N}(\rho = 0) = 1$ and $\partial\mathcal{N}/\partial\rho|_{\rho=0} = 0$. Take

$$\mathcal{N} = 1 + \delta\mathcal{N} \qquad (11)$$

and expand Eq. (10) to leading order in $\delta\mathcal{N}$ to obtain

$$\frac{\partial^2\delta\mathcal{N}}{\partial\rho^2} \simeq (1 - 2\Delta)\delta\mathcal{N} - \Delta \qquad (12)$$

subject to $\delta\mathcal{N}(\rho = 0) = 0 = \partial\delta\mathcal{N}/\partial\rho|_{\rho=0}$. The solution is

$$\delta\mathcal{N} = -\frac{\Delta}{2}\rho^2. \qquad (13)$$

It is easy to see from Eqs. (11) and (13) how precision must be carefully tracked for very small Δ to produce a valid numerical solution for $\psi = -\ln\mathcal{N}$ in Eq. (6) with limited machine/method precision. Choose a cut-off value of ρ, $\rho = \rho_c$ where the solution for $\delta\mathcal{N}$ in Eq. (13) will be applied for $\rho \in [0, \rho_c]$ and denote $\delta\mathcal{N}(\rho = \rho_c) = -(\Delta/2)\rho_c^2 \equiv -\delta\mathcal{N}_c$. We use δN_c as a control parameter to set ρ_c and take

$$\psi = \begin{cases} -\ln\left(1 - \frac{\Delta}{2}\rho^2\right), & 0 \le \rho \le \sqrt{\frac{2\delta\mathcal{N}_c}{\Delta}}, \\ \text{Numerical}, & \sqrt{\frac{2\delta\mathcal{N}_c}{\Delta}} \le \rho \le \rho_{\max}. \end{cases} \qquad (14)$$

Here, numerical solution is constructed by integrating Eq. (6) for $\rho \in [\rho_c, \rho_{\max}]$ subject to the initial condition

$$\psi(\rho = \rho_c) = -\ln\left(1 - \frac{\Delta}{2}\rho_c^2\right), \quad \left.\frac{\partial\psi}{\partial\rho}\right|_{\rho=\rho_c} = \frac{\Delta\rho_c}{1 - \Delta\rho_c^2/2},$$

and $\rho = \rho_{\max}$ is some maximum value of ρ where $\mathcal{N}(\rho = \rho_{\max})$ is negligible. This value can be found numerically. From numerical tests we find for small Δ that the flat central region of $\mathcal{N}$ extends from $\rho = 0$ to $\rho = \rho_{\text{flat}} \simeq 2.3\log_{10}(\Delta)$. Taking $\rho_{\max} = \rho_{\text{flat}} + \rho_{\text{edge}}$ where $\rho_{\text{edge}} \simeq 5$ to account for the (invariant of scale of Δ) fall-off of the edge of $\mathcal{N}$ is adequate to place $\rho_{\max}$ where $\mathcal{N}$ is exponentially small, but not so large to result in unnecessary numerical work. For numerical integration methods with fractional numerical precision 10^{-p}, it should be safe to choose the cutoff parameter as $\delta\mathcal{N}_c \sim 10^{-(p-2)}$. Direct numerical integrations of Eq. (6) for ψ works for $\Delta \sim> 10^{-4}$ and the procedure outlined above should (unless high precision integrators are employed) be applied for $\Delta \sim< 10^{-4}$.

DISCUSSION

Details have been provided on a simple but robust numerical method to solve the thermal equilibrium Poisson equation for a 1D sheet beam for arbitrarily strong space-charge intensity. This method was extensively employed in Ref. [1] and can be applied in slightly modified form to higher dimensions. Similar techniques can be applied to other, non-thermal choices of smooth $f(H)$ representing continuously focused equilibrium distributions both for 1D sheet beam models and in higher dimensions.

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