A NEW APPROACH TO CALCULATE THE TRANSPORT MATRIX IN RF CAVITIES*

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Abstract

A realistic approach to calculate the transport matrix in RF cavities is developed. It is based on joint solution of equations of longitudinal and transverse motion of a charged particle in an electromagnetic field of the linac. This field is a given by distribution (measured or calculated) of the longitudinal electric field on the axis of the linac. This new approach is compared with other matrix methods to solve the same problem. The comparison with code ASTRA has been carried out. A complete agreement for tracking results for a TESLA-type cavity is achieved. A corresponding algorithm will be implemented into the MARS15 code.

TRANSVERSE MOTION OF THE CHARGE PARTICLE IN THE RF CAVITY

Let us consider a cavity with the given distribution of longitudinal field along its axis, so that the function $E_z(z)$ is known. Other components, radial electrical field $E_r(z)$ and azimuthal magnetic one $H_\varphi(z)$ can be expressed according to the following Maxwell's equations:

$$\vec{E}_r(z) = -\frac{r}{2}\frac{\partial E_z(z)}{\partial z}\vec{e}_r \text{ and } \vec{B}_\varphi(z) = \frac{r}{2c}\frac{\partial E_z(z)}{\partial t}\vec{e}_\varphi = \frac{\beta r}{2}\frac{\partial E_z(z)}{\partial z}\vec{e}_\varphi,$$

where r is the distance from the cavity axis and β is the relative velocity of the particle. These transverse fields result in a beam focusing due to Lorenz force:

$$\vec{F}_r = -q(E_r - \beta B_\varphi)\vec{e}_r = \frac{q(1+\beta^2)}{2} r \frac{\partial E_z}{\partial z}\vec{e}_r, \quad (1)$$

where q, m are charge and mass of the particle. In the case of an axially-symmetric cavity the following equation for transverse motion of the particle can be obtained from expression (1):

$$x'' + \frac{1}{\beta^2}\frac{\gamma'}{\gamma}x' = \frac{q}{mc^2}\frac{1+\beta^2}{2\beta^2\gamma}x\frac{\partial E_z}{\partial z}. \quad (2)$$

Here the prime means derivative with respect to z. Usually, the motion of a particle is considered as ultra relativistic, so that $\left(1+\beta^2\right)/2\beta^2 \approx 1$, but we will not restrict ourselves to this case only.

The standard approach [1, 2] is as follows. The electromagnetic field of an RF cavity includes a few higher spatial (temporal) harmonics. For this reason it is possible to present the motion of the charge particle as sum of two components: smooth ("slow") and "fast" and apply the matrix approach to solve the equation (2) . After averaging over time, significantly longer than characteristic time of the fast component, and after the necessary transformations, it is possible to reduce the equation (2) to the following form:

$$x''(z) + \frac{1}{\beta^2}\frac{\gamma'}{\gamma}x'(z) + \frac{1}{2}\left(\frac{qE_0}{mc^2}\frac{1+\beta^2}{4\beta^2\gamma}\right)^2 x(z)\eta(\phi) = 0. \quad (3)$$

Some special factor $\eta(\phi)$ appears [1,2] in this equation to takes into account an RF-field structure in the cavity and particle's phase head of crest, ϕ .

The solution of equation for an ultra relativistic particle in the case of the "pure" (without other spatial harmonics) $\pi -$mode of the field in the cavity can be written using the matrix obtained Chambers [3, 4], which already takes into account the effect of the edges in transverse focusing [1, 5] at cavity entrance/exit:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_f = M^{Ch}\begin{pmatrix} x \\ x' \end{pmatrix}_i \quad (4)$$

The components of matrix Chambers M^{Ch} are:

$$\left.\begin{aligned} M_{11}^{Ch} &= \cos\alpha - \sqrt{2}\cdot\cos\phi\cdot\sin\alpha; \\ M_{12}^{Ch} &= \sqrt{8}\frac{\gamma_i}{\gamma'}\cdot\cos\phi\cdot\sin\alpha; \\ M_{21}^{Ch} &= -\left(\frac{\gamma'}{\gamma_f}\right)\cdot\left[\frac{\cos\phi}{\sqrt{2}} + \frac{1}{\sqrt{8}\cos\phi}\right]\cdot\sin\alpha; \\ M_{22}^{Ch} &= \left(\frac{\gamma_i}{\gamma_f}\right)\cdot\left[\cos\alpha + \sqrt{2}\cdot\cos\phi\cdot\sin\alpha\right], \end{aligned}\right\} \quad (5)$$

where $\alpha = \frac{1}{\sqrt{8}\cos\phi}\ln\frac{\gamma_f}{\gamma_i}$.

Some other matrix representations of the particle motion in the cavity are discussed in details in Ref. [6].

NEW MATRIX APPROACH

The main feature of this approach is to use the known distribution of the electric field at the cavity axis $E_z(z,t)$, and the paraxial approximation for the particle

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motion. In a general case we are not considering ultra relativistic energies of the beam

Equation and Solution

The equation for the longitudinal motion (acceleration) of a particle in the RF field is the following:

$$\frac{d\gamma}{dt}=\frac{q\beta}{mc}E_z(z,t). \qquad (6)$$

The equation for transverse motion can be rewritten from (2) in more convenient form [6]:

$$m\gamma\ddot{x}+\frac{q\beta}{c}E_z\dot{x}+\frac{q}{2}\left(\frac{\partial E_z}{\partial z}+\frac{\beta}{c}\frac{\partial E_z}{\partial t}\right)x=0. \qquad (7)$$

For an RF cavity with the transverse TM-mode

$$E_z(z,t)=E_0(z)\cos\left(\omega t+\phi\right) \qquad (8)$$

the equation (7) takes the form

$$m\gamma\ddot{x}+\frac{q\beta}{c}E_0(z)\cos(\omega t+\phi)\dot{x}+$$
$$\frac{q}{2}\left[\frac{\partial E_0}{\partial z}\cos(\omega t+\phi)-\frac{\omega\beta}{c}E_0(z)\sin(\omega t+\phi)\right]x=0$$

and the equation (6) can be integrated, so that relative gain $\Delta\gamma/\gamma$ of the particle energy while moving through the cavity during the time interval $t\div t+\Delta t$ is equal to

$$\frac{\Delta\gamma}{\gamma}=\frac{q\bar{\beta}\bar{E}_0}{mc\bar{\gamma}}\cdot\Delta t\cdot\frac{\sin(\omega\cdot\Delta t/2)}{\omega\cdot\Delta t/2}\cdot\cos(\omega\bar{t}+\phi). \qquad (9)$$

All "bar-values" in this expression are referred to the moment $t+\Delta t/2$. It is convenient to use a "length" $\tau=ct$, wave number $k_0=w/c$ and express the amplitude of the electric field as $\tilde{E}_0=qE_0/mc^2$. Then one can find the final equation for the transfer motion (a sign "'" means now the derivative with respect to length, τ):

$$x''+\frac{\beta}{\gamma}\tilde{E}_0(z)\cos(k_0\tau+\phi)x'+$$
$$\frac{1}{2\gamma}\left[\frac{\partial\tilde{E}_0}{\partial z}\cos(k_0\tau+\phi)-k_0\beta\tilde{E}_0(z)\sin(k_0\tau+\phi)\right]x=0. \qquad (10)$$

To integrate this equation over τ it is necessary to present the whole integration range as a number of subintervals (slices) $\tau\div\tau+\Delta\tau$ and corresponding slices $z\div z+\Delta z$. On each of these subintervals one can neglect a change of parameters $\beta,\ \gamma$ as well as of the values of the field $\tilde{E}_0(z)$ and its derivative. In this approach, instead of equation (10) one has the simplest equation of the second order with "constant" coefficients:

$$x''+zx'+bx=0, \qquad (11)$$

where

$$a=\frac{\bar{\beta}}{\bar{\gamma}}\tilde{E}_0(\bar{z})\cos(k_0\bar{\tau}+\phi),$$
$$b=\frac{1}{2\bar{\gamma}}\left[\frac{\partial\tilde{E}_0(\bar{z})}{\partial z}\cos(k_0\bar{\tau}+\phi)-k_0\bar{\beta}\tilde{E}_0(\bar{z})\sin(k_0\bar{\tau}+\phi)\right]$$

with $\bar{\beta}=\beta(\bar{z})$, $\bar{\gamma}=\gamma(\bar{z})$ and $\bar{\tau},\ \bar{z}$ are the centers of the slices. It is quite easy to find a solution to this equation for the coordinate x_f and angle x_f' at the exit of the cavity using their values $x_i,\ x_i'$ at the entrance:

$$\left.\begin{aligned}x_f(\tau)&=\frac{-\alpha_2x_i+x_i'}{\alpha_1-\alpha_2}e^{\alpha_1\tau}+\frac{\alpha_1x_i-x_i'}{\alpha_1-\alpha_2}e^{\alpha_2\tau},\\ x_f'(\tau)&=\frac{-\alpha_2x_i+x_i'}{\alpha_1-\alpha_2}\alpha_1e^{\alpha_1\tau}+\frac{\alpha_1x_i-x_i'}{\alpha_1-\alpha_2}\alpha_2e^{\alpha_2\tau}\end{aligned}\right\},$$

where . $\alpha_{1,2}=(-a\pm\sqrt{a^2-4b})/2$ These expressions allow one to find the desired matrix of the transformation of the coordinate vector during particle passage through the cavity. Let us use the coefficients $\alpha_{1,2}$ and introduce the following parameters:

$$\left.\begin{aligned}\varepsilon^2&=\frac{1}{2\bar{\gamma}}\Big[\bar{\beta}^2\tilde{E}_0^2(\bar{z})\cos^2(k_0\bar{\tau}+\phi)-\frac{\partial\tilde{E}_0(\bar{z})}{\partial z}\cos(k_0\bar{\tau}+\phi)+\\&\qquad k_0\bar{\beta}\tilde{E}_0(\bar{z})\sin(k_0\bar{\tau}+\phi)\Big];\\ \delta&=\frac{\bar{\beta}}{2\bar{\gamma}}\tilde{E}_0(\bar{z})\cos(k_0\bar{\tau}+\phi),\end{aligned}\right\}$$

So, $\alpha_{1,2}=-\delta\pm\varepsilon$ and after simple manipulations the following result will be found for the matrix M of the slice of the cavity with length $\Delta\tau$:

$$\left.\begin{aligned}M_{12}&=e^{-\delta\cdot\Delta\tau}\cdot\frac{\sinh(\varepsilon\cdot\Delta\tau)}{\varepsilon},\\ M_{11}&=e^{-\delta\cdot\Delta\tau}\cdot\cosh(\delta\cdot\Delta\tau)+M_{12}\cdot\delta,\\ M_{21}&=(\varepsilon^2-\delta^2)\cdot M_{12},\\ M_{22}&=M_{11}-2\cdot M_{12}\cdot\delta.\end{aligned}\right\} \qquad (12)$$

It is is very simple to calculate the determinant of the transport matrix M of the cavity: $\det M=e^{-2\delta\cdot\Delta\tau}=e^{-\Delta\gamma/\gamma}$. As mentioned above, during slicing of the whole cavity in order to integrate the equation of transverse motion of the particle it is necessary to take into account that for each slice the relative acceleration rate must be small, i.e. for all subintervals with length $\Delta\tau$ the value $\Delta\gamma/\gamma \ll 1$, so it is possible to replace the direct integration by a solution which uses the matrix approach.

Verification of the New Approach

To verify this approach, the code MatLab Dark Current (MLDC) was created. This code realizes two possibilities: direct integration of equation (11) by method Runge-Kutta with a fixed time step (4th order; function ode45 from the MatLab package) and matrix approach (using expressions(12)) for this equation. To calculate the acceleration rate the expression (9) was used.

The results of simulations with the code MLDC were compared with the results (naturally, for the same input data), received while using the code ASTRA (A Space charge TRacking Algorithm) [7].

To compare both codes, the TESLA-type cavity is used with field amplitude $E_0 = 36.815 MV/m$ MV/m.

Results of the scanning over a phase for both codes are shown in Fig. 1.

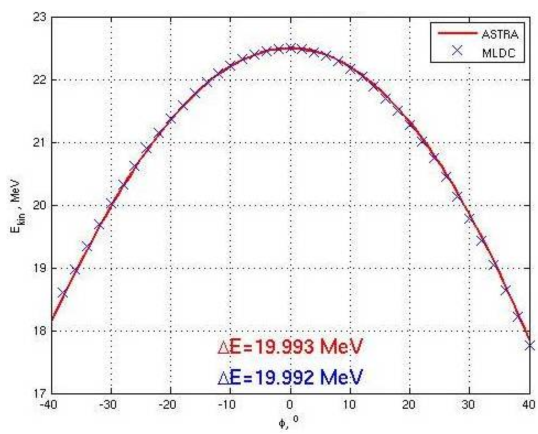

Figure 1: Energy gain in the Tesla-type cavity depending on phase of the particle.

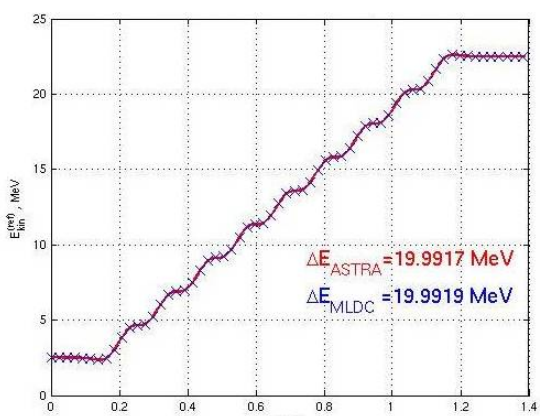

Figure. 2: Acceleration in the Tesla-type cavity.

The Fig. 2 is illustrated the process of the acceleration and gives the same results for both codes.

Next figures demonstrate the result of tracking with both codes in the cases of DI (direct integration) and MA (matrix approach).

A complete agreement between all results is achieved. It proves the validity of the code MLDC and approaches used to create it.

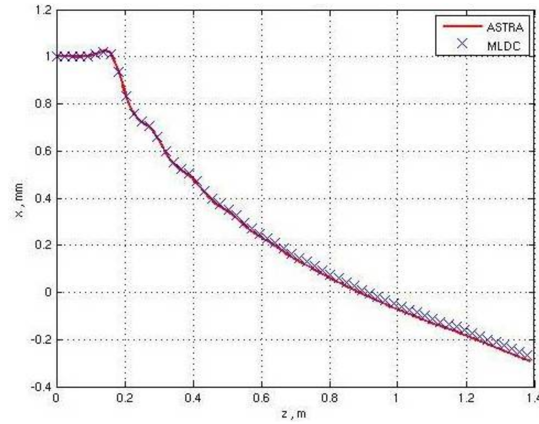

Figure 3: Particle's track (DI approach). $E_0 = 2.5MeV$.

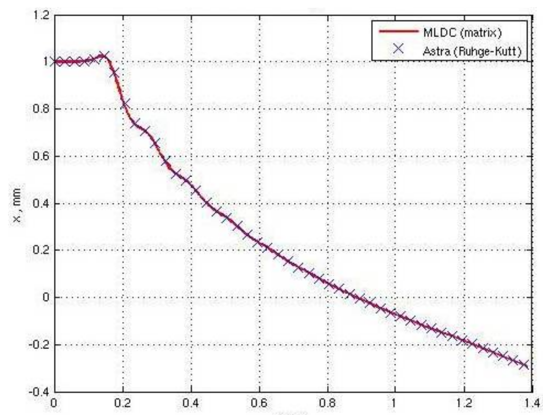

Figure 4: Particle's track (MA approach). $E_0 = 2.5MeV$.

CONCLUSIONS

A realistic approach to calculate the transport matrix in RF cavities is developed. Complete agreement for tracking results with existed code ASTRA is achieved. New algorithm will be implemented into MARS15 code.

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