MAGNETIC FIELD EXPANSION OUT OF A PLANE: APPLICATION TO CYCLOTRON DEVELOPMENT*

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Abstract

In studies of the dynamics of charged particles in a cyclotron magnetic field, the specified field is generally B_z in the $z = 0$ midplane where B_r and B_θ are zero. $B_r(r, \theta, z)$ and $B_\theta(r, \theta, z)$ are sometimes determined through a linear expansion which assumes that B_z is independent of z. An expansion to only first order may not be sufficient for orbit simulations at large z. This paper reviews the expansion of a specified $B_z(r, \theta, z = 0)$ out of the $z = 0$ midplane to arbitrary order, and shows an application to a simulated inverse cyclotron for muon cooling.

INTRODUCTION

A number of programs and software packages simulate cyclotron magnetic fields and orbits of charged particles through such fields. An important preliminary step before simulation of fields produced by magnets is investigation of orbit dynamics in an idealized magnetic field which satisfies Maxwell's equations in vacuum to sufficient order. Magnetic fields for cyclotrons [1] are most conveniently expressed in cylindrical (r, θ, z) coordinates in which the z direction is the cyclotron rotation axis. In many applications, an ideal field is specfied in the cyclotron median plane which is the $z = 0$ plane where B_r and B_θ are zero. Determination of (B_r,B_θ,B_z) for z $\neq$ 0 to sufficient order (beyond 1st order) is necessary for detailed orbit and dynamics studies. Reduction of the phase space of a muon beam through ionization cooling in an inverse cycloton is being researched [2, 3, 4] as part of the muon R&D for a future neutrino factory ($\mu^- \to e^- \,\overline{\nu}_e\, \nu_\mu$ and $\mu^+ \to e^+\, \nu_e\, \overline{\nu}_\mu$) and energy frontier muon collider [5].

FIELD EXPANSION OUT OF MIDPLANE

The importance of correct field expansion for reliable particle tracking simulation and beam dynamics analysis has been recognized, and the expansion around the reference orbit of a bent-solenoid geometry has been calculated [6]. Expansions to 8th order are done with the aid of Mathematica. A static magnetic field can be expanded out of a plane because it must satisfy the source-free Maxwell equations in vacuum, $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$. Only $\partial B_x/\partial y = \partial B_y/\partial x$ constrains fields in the plane, so any function can be used for B_z. One constrains the three field components out of the plane as follows:

$$\frac{\partial B_z}{\partial z} = -\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y} \qquad \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} \qquad \frac{\partial B_y}{\partial z} = \frac{\partial B_z}{\partial y}$$

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The expansion of a magnetic field specified in a plane has been addressed by Zgoubi for cartesian coordinates [7]. Zgoubi uses $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{B} = 0$, and Taylor expansion. Here we work out a solution in cylindrical coordinates to arbitrary order. Median plane antisymmetry is assumed so that

$$B_r(r, \theta, 0) = 0 \qquad B_\theta(r, \theta, 0) = 0$$

$$B_r(r, \theta, z) = -B_r(r, \theta, -z)$$

$$B_\theta(r, \theta, z) = -B_\theta(r, \theta, -z)$$

$$B_z(r, \theta, z) = B_z(r, \theta, -z)$$

With $B \equiv B_z(r, \theta, z = 0)$ the magnetic field is

$$B_r = z\frac{\partial B}{\partial r} - \frac{z^3}{6}\frac{\partial}{\partial r}\left(\frac{\partial^2 B}{\partial r^2} + \frac{1}{r}\frac{\partial B}{\partial r} + \frac{1}{r^2}\frac{\partial^2 B}{\partial \theta^2}\right) + \ldots$$

$$B_\theta = \frac{z}{r}\frac{\partial B}{\partial \theta} - \frac{z^3}{6}\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial^2 B}{\partial r^2} + \frac{1}{r}\frac{\partial B}{\partial r} + \frac{1}{r^2}\frac{\partial^2 B}{\partial \theta^2}\right) + \ldots$$

$$B_z = B - \frac{z^2}{2}\left(\frac{\partial^2 B}{\partial r^2} + \frac{1}{r}\frac{\partial B}{\partial r} + \frac{1}{r^2}\frac{\partial^2 B}{\partial \theta^2}\right) + \ldots$$

Defining $\nabla^2_{r,\theta} \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$,

$$B_r = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}\frac{\partial}{\partial r}\left[\left(\nabla^2_{r,\theta}\right)^n B\right]$$

$$B_\theta = \frac{1}{r}\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}\frac{\partial}{\partial \theta}\left[\left(\nabla^2_{r,\theta}\right)^n B\right]$$

$$B_z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}\left(\nabla^2_{r,\theta}\right)^n B$$

A rigorous proof of these expansions is provided in [9].

In most implementations of simulating a magnetic field, the expansions of (B_r, B_θ, B_z) cannot be expressed in closed form. In such cases, only a finite number of terms of the magnetic field expansion can be used. The order of the expansion is defined as the highest power of z in the summation. To odd (even) order m, B_r and B_θ have $\frac{m+1}{2}$ ($\frac{m}{2}$) terms while B_z has $\frac{m+1}{2}$ ($\frac{m+2}{2}$) terms. When $\mathbf{B}$ is expanded to odd order, $\nabla \times \mathbf{B} = 0$ and $\nabla \cdot \mathbf{B} \neq 0$; when $\mathbf{B}$ is expanded to even order, $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} \neq 0$. An even order expansion with $\nabla \cdot \mathbf{B} = 0$ is needed to satisfy the Hamiltonian and allow a symplectic map.

Here is a 4th order magnetic field expansion done with the aid of Mathematica for an N sector cyclotron. The field goes like $B_z = B_0(1 - f\sin(N\theta))$ in the midplane and is constant with radius. The spiral angle is zero.

$$B_r = \frac{1}{3}\left(\frac{z}{r}\right)^3 (fB_0N^2\sin(N\theta))$$

$$B_\theta = -\frac{z}{r}(fB_0N\cos(N\theta)) - \frac{1}{6}\left(\frac{z}{r}\right)^3 (fB_0N^3\cos(N\theta))$$

$$B_z = B_0(1 - f\sin(N\theta)) - \frac{1}{2}\left(\frac{z}{r}\right)^2 (fB_0N^2\sin(N\theta)) + \frac{1}{24}\left(\frac{z}{r}\right)^4 (fB_0N^2\sin(N\theta))(4 - N^2)$$

AN INVERSE SECTORED CYCLOTRON

A large admittance, inverse sector cyclotron might reduce the phase space of a muon beam for use in a ν factory or $\mu^+\mu^-$ collider. An azimuthally sectored field can provide strong axial focusing and a spiral field even stronger focusing. Increasing the magnitude of $B_z(r,\theta,z=0)$ with r provides more radial focusing but adds some axial defocusing. In an inverse cyclotron, the outer sectored field must transition to an inner, azimuthally symmetric magnetic bottle. The lattice forms a quasipotential well to ferry muons into the bottle as they lose angular momentum.

Simulations of this muon inverse cyclotron are done using G4Beamline [8]. Muons are cooled in a 3 step process:

1. Positive muons with an average momentum of 180 MeV/c are slowed down by six lithium hydride wedges, one in each cyclotron valley, and stop in 0.1 bar helium near $r = 0$. Low Z minimizes multiple scattering. Each LiH wedge decreases adiabatically from a thickness of 4.2 mm at $r = 550$ mm to 0.008 mm at $r = 65$ mm. Thickness is set by the $dE/dx \sim \beta^{-5/3}$ slope. Helium has a lower cross section for μ^+e^- formation than hydrogen. Single turn, energy loss injection has been simulated previously [2].

2. Stopped muons are extracted axially through a 200 mm high helium column by an 0.1 MV/m electric field after the whole muon bunch train has arrived. A few hundred nanosecond time spread occurs because muons travel through different distances of helium.

3. Extracted muons are accelerated in vacuum from below the ionization energy loss peak at 8 keV (permits frictional cooling) to 180 MeV/c (KE$\sim$100 MeV) by a 100 m long, 1 MV/m electric field in a solenoid.

The midplane B field used for the outer sectors has the form and parameters in Table 1. The inner bottle field is generated by two $r = 0.2$ m coils, located 0.2 m above and below the midplane. The B field at $r = 0$ is 2.4 T (Fig. 1).

A beam of 200 muons with spreads in initial positions, angles, times, and momenta was sent through the inverse cyclotron. Fig. 2 shows the orbit of one of the 14 muons which stopped near the middle, and Fig. 3 shows the distance from the midplane vs. time for the same muon.

Table 1: Outer Focus Field $B_z(z=0)$ in $z=0$ midplane

$c\,r^k(1+f\cos(N(\theta-\tan(\xi)\ln\frac{r}{r_0})))\frac{1+\tanh(b(r-r_1))}{2}$	
c	1.77 T m$^{-0.6}$
k - field index	0.6
f - flutter amplitude	$\sqrt{2}$
N - number of sectors	6
ξ - spiral angle	0°
r_0	0.01 m
b - strength of transition	25 m^{-1}
r_1 - location of transition	0.2 m

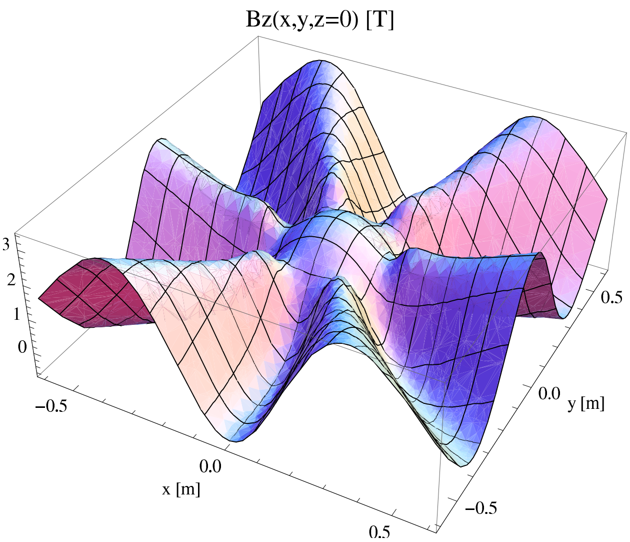

Figure 1: $B_z(x,y,z=0)$ including the outer 6-sector focus field and the inner magnetic bottle. The outer focus field is generated by a specified $B_z(r,\theta,z=0)$ which is expanded to 2nd order out of the $z=0$ midplane. Fields range from - 0.6 to 3.2 Tesla. The inner bottle field is generated through two current carrying circular coils.

The 14 stopped muons were extracted from the helium and then accelerated to 180 MeV/c after traveling 100 m along z. The initial and final normalized emittances of the cooled muons are shown in Table 2, indicating modest cooling in the transverse directions (about 4.5-fold emittance reduction in each direction) and strong longitudinal cooling (about 88,000-fold). Fig. 4 shows the initial and final energy and times spreads.

Needed improvements to make the design realistic include: (1) reducing the spiral time and muon decay losses by decreasing the inner magnetic field so lower momentum muons pass through thicker LiH wedges, (2) increasing the admittance into the bottle, (3) reducing large exit cone losses by adding an electric field to turn the magnetic bottle into a Penning trap, (4) replacing the helium with a very thin foil as is done at the PSI inverse μ^- cyclotron [3] thus

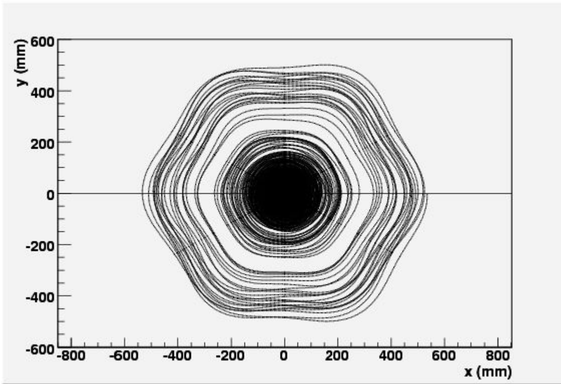

Figure 2: Orbit of one of the stopped muons.

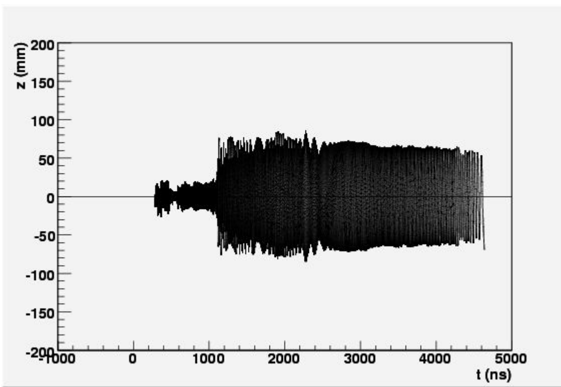

Figure 3: z vs. t for one of the stopped muons. The muon entered the magnetic bottle at $t = 1100$ ns.

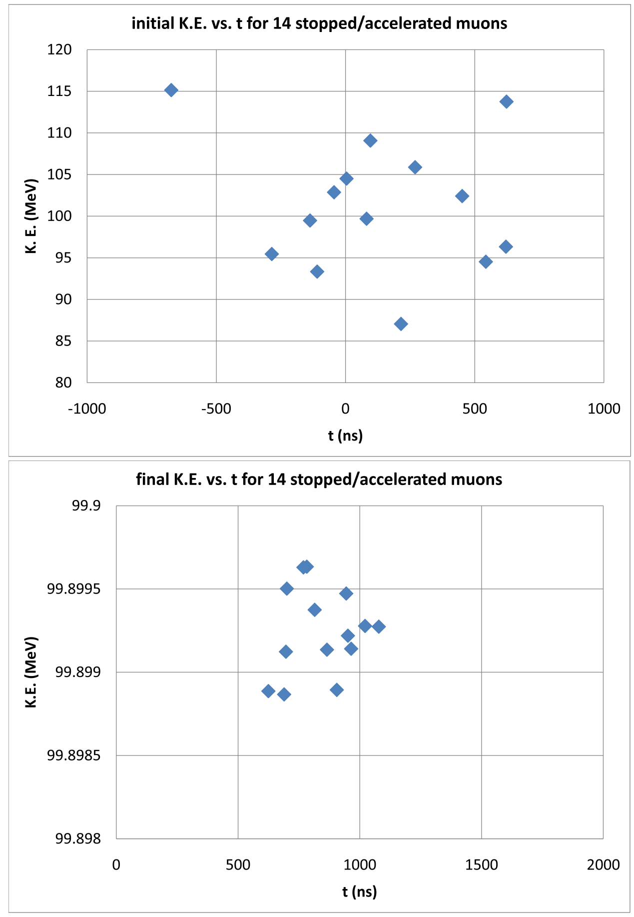

Figure 4: Initial and final kinetic energy vs. time phase space plots for 14 cooled muons. Extraction starts at $t = 0$.

trapping muons in vacuum to reduce ionization buildup, muonium formation, and nuclear capture, and (5) designing a skew quadrupole triplet to transform the extracted round spinning beam into a flat non-spinning beam [10]. We might transition the cyclotron from six to three sectors as radius decreases. The Brillouin space charge limit from plasma physics, $\epsilon_0 B^2/2m_\mu$, indicates that a 1 Tesla magnetic field can trap 2×10^{12} muons in 85 cc.

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Table 2: Initial and Final Normalized Emitances

Direction	Initial ϵ_N (mm rad)	Final ϵ_N (mm rad)
Transverse (1)	0.12	0.026
Transverse (2)	0.11	0.026
Longitudinal	8300	0.094

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