TRANSVERSE BEAM TRANSFER FUNCTIONS VIA THE VLASOV EQUATION*

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Abstract

A semi-numerical method of integrating the Vlasov equation to obtain beam transfer functions directly as a function of frequency is presented. The results are compared with beam transfer functions calculated via particle tracking and excellent agreement is shown. The technique works well with both transverse wakes and detuning wakes from space charge.

INTRODUCTION AND THEORY

The stability and response properties of particle beams are encapsulated in beam transfer functions (BTFs) [1, 2, 3]. BTFs are relatively easy to measure making their accurate calculation of great practical interest. In this paper we will present two independent ways to calculate bunched beam transverse beam transfer functions, and show their agreement for some challenging parameters. To begin let θ denote azimthh, which increases by 2π each turn. Let t be clock time and let τ denote particle arrival time with respect to the synchronous particle so that $\theta = \omega_0(t-\tau)$, with ω_0 the synchronous angular revolution frequency. We use θ as the time-like variable and, to make a precise cross check possible, take linear rf so that

$$\frac{d\tau}{d\theta} = Q_s\epsilon \qquad \frac{d\epsilon}{d\theta} = -Q_s\tau, \tag{1}$$

where Q_s is the synchrotron tune and ϵ is the energy variable. with We consider a single transverse variable x and uniform focusing. Collective forces are taken in the continuum approximation with $dx/d\theta = p$ and

$$\begin{aligned}\frac{dp}{d\theta} &= -Q_\beta^2(\epsilon)x + F_e(\theta,\tau)\\ &+ 2Q_\beta\Delta Q_{sc}(\tau)[x-\bar{x}(\theta,\tau)]\\ &- \frac{q}{2\pi P_0\omega_0}\int_0^{2\tau_b} W_\perp(\tau_1)D(\theta,\tau-\tau_1)d\tau_1,\end{aligned} \tag{2}$$

where $Q_\beta(\epsilon) = Q_0 + Q_s\epsilon\omega_0\xi/\eta$ is the betatron tune, with chromaticity ξ and frequency slip factor η, $F_e(\theta,\tau)$ is the external driving force, $\Delta Q_{sc}(\tau)$ is the space charge tune shift as a function of longitudinal position in the bunch, and $\bar{x}(\theta,\tau)$ is the average beam offset. The wall induced forces are due the the transverse wake potential $W_\perp(\tau)$ and driven by the instantaneous dipole moment $D(\theta,\tau) = \bar{x}(\theta,\tau)I(\tau)$ with $I(\tau)$ the bunch current, q the particle charge, and P_0 the synchronous momentum. The force is limited to a single bunch of full length $2\tau_b$ but can be extended to uniformly filled rings.

To solve (1) and (2) using the Vlasov equation first introduce amplitude angle variables defined by $\tau = a\sin\psi$ $\epsilon = a\cos\psi$. Since all particle have constant a,

$$\frac{\partial F}{\partial\theta} + p\frac{\partial F}{\partial x} + \frac{dp}{d\theta}\frac{\partial F}{\partial p} + Q_s\frac{\partial F}{\partial\psi} = 0, \tag{3}$$

where $dp/d\theta$ is given by Eq (2). We normalise F so that $F dxdpadad\psi$ is the number of particles in the phase space volume. To continue we define 3 transverse moments [4],

$$\{X(\psi,a,\theta), P, \Psi\} = \int dxdpF(x,p,a,\psi,\theta)\{x,p,1\}. \tag{4}$$

This gives

$$\frac{\partial X}{\partial\theta} + Q_s\frac{\partial X}{\partial\psi} = P(\psi,a,\theta), \tag{5}$$

$$\begin{aligned}\frac{\partial P}{\partial\theta} + Q_s\frac{\partial P}{\partial\psi} &= -Q_\beta^2(\epsilon)X + F_e(\theta,\tau)\Psi\\ &+2Q_\beta\kappa\left[X\int d\epsilon\Psi - \Psi\int d\epsilon X\right]\\ &-\frac{q\Psi}{2\pi P_0\omega_0}\int_0^\infty d\tau_1 W_\perp(\tau_1)q\int d\epsilon X(\epsilon,\tau-\tau_1)\end{aligned} \tag{6}$$

where $\Delta Q_{sc}(\tau) = \kappa\int d\epsilon\Psi$, $\bar{x}(\tau)\int d\epsilon\Psi = \int d\epsilon X$ and, for example,

$$\int d\epsilon X \equiv \int a_1da_1d\psi_1\delta(a\sin\psi - a_1\sin\psi_1)X(\psi_1,a_1,\theta),$$

and the occurences of τ and ϵ in (6) are understood to be shorthand for the amplitude angle representations. Next we substitute (5) in (6) and drop second partial derivatives with respect to ψ. We take $F_e(\theta,\tau) = F_0(\tau)\exp(-iQ\theta)$, and we take $X = X_1\exp(-iQ\theta - i\xi\omega_0\tau/\eta)$, with $Q = Q_0 + \Delta Q$. This yields

$$\left\{\Delta Q + iQ_s\frac{\partial}{\partial\psi} + \Delta Q_{sc}(a\sin\psi)\right\}X_1(\psi,a) = \tilde{F}(\tau)\Psi(a) \tag{7}$$

where

$$\begin{aligned}\tilde{F}(\tau) &= \frac{-e^{i\xi\omega_0\tau/\eta}F_0(\tau)}{2Q_0} + \kappa\int d\epsilon X_1\\ &+q^2\int_0^\infty d\tau_1\frac{W_\perp(\tau_1)e^{i\xi\omega_0\tau_1/\eta}}{4\pi Q_0P_0\omega_0}\int d\epsilon X_1(\epsilon,\tau-\tau_1).\end{aligned} \tag{8}$$

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The strategy is to solve (7) for $D_1 = q\int d\epsilon X_1$ resulting in a one dimensional integral equation for D_1 which will be solved numerically. To proceed let $\Delta Q_{sc}(a\sin\psi) = \lambda(a) + d\Lambda/d\psi$ where $\Lambda(a,\psi) = \Lambda(a,\psi+2\pi)$. Substitute $X_1 = X_2\exp(i\Lambda/Q_s)$ yielding

$$\left\{\Delta Q + iQ_s\frac{\partial}{\partial\psi} + \lambda(a)\right\} X_2 = e^{-i\Lambda/Q_s}\tilde{F}\Psi. \quad (9)$$

Multiply both sides by the integrating factor $\exp(-i\psi(\lambda+\Delta Q)/Q_s)$ and integrate from ψ to $\psi+2\pi$ employing the periodicity of X_2. Backsubstitute X_1 giving

$$X_1 = \frac{\Psi e^{iG(a,\psi)}}{Q_s R(a)}\int_{\psi}^{\psi+2\pi} d\psi_1\tilde{F}(a\sin\psi_1)e^{-iG(a,\psi_1)} \quad (10)$$

where $R(a) = 1-\exp(-2\pi i(\lambda+\Delta Q)/Q_s)$ and

$$G(\tau,\psi) = \int_0^{\psi} d\psi_1\frac{\Delta Q + \Delta Q_{sc}(a\sin\psi_1)}{Q_s}.$$

To proceed note that

$$D_1(\tau) = q\int adad\psi\delta(\tau - a\sin\psi)X_1(a,\psi)$$

yielding

$$D_1(\tau) = \int\hat{G}(\tau,\tau_1)q\tilde{F}(\tau_1)d\tau_1 \quad (11)$$

where

$$q\tilde{F}(\tau) = \frac{-e^{i\xi\omega_0\tau/\eta}qF_0(\tau)}{2Q_0} + \kappa D_1(\tau)$$
$$+\frac{q^2}{4\pi Q_0P_0\omega_0}\int_0^{\infty} d\tau_1 W_\perp(\tau_1)e^{i\xi\omega_0\tau_1/\eta}D_1(\tau-\tau_1) \quad (12)$$

and

$$\hat{G}(\tau,\tau_1) = \int_0^{\infty} ada\int_0^{2\pi} d\psi\delta(\tau - a\sin\psi)\Psi(a)e^{iG(a,\psi)}$$
$$K(a)\int_{\psi}^{\psi+2\pi} d\psi_1\delta(\tau_1 - a\sin\psi_1)e^{-iG(a,\psi_1)} \quad (13)$$

where $K = i\Psi/Q_sR$. To proceed assume $D_1(\tau)$ is defined at a set of equidistant points $\tau_k = k\Delta$ and that D varies linearly between these points. Then

$$q\tilde{F}(\tau) = \sum_{k=-N}^{N} a_kT(\tau - k\Delta),$$

where $T(x)$ is a triangle function of height 1 and half width at base Δ. For smooth bunches $D(\tau)$ is zero at the ends of the bunch so we take $\Delta = \tau_b/(N+1)$ where τ_b is the half bunch length. Next we do the integral over ψ in (13). For any smooth function $h(a,\psi)$,

$$\int_0^{2\pi}\delta(\tau - a\sin\psi)h(a,\psi)d\psi = \sum_{p=1,2}\frac{h(a,\psi_p)}{\sqrt{a^2-\tau^2}},$$

where the two angles satisfy $\sin\psi_{1,2} = \tau/a$ and the integral vanishes for $|\tau| > a$. Next we define a new radial variable u defined by $u^2 = a^2 - \tau^2$ so $udu = ada$. The dipole density is defined at the same set of lattice points as $\tilde{F}$ so

$$D_1(m\Delta) \equiv D_m = \sum_{n=-N}^{N} a_nM_{m,n}, \quad (14)$$

where

$$M_{m,n} = \int_0^{\sqrt{\tau_b^2 - m^2\Delta^2}} duK(a)\sum_{p=1,2} e^{iG(a,\psi_p)}$$
$$\int_{\psi_p}^{\psi_p+2\pi} d\psi_1 T(n\Delta - a\sin\psi_1)e^{-iG(a,\psi_1)}. \quad (15)$$

In integral (15) $a = \sqrt{u^2+m^2\Delta^2}$ and $\sin\psi_p = m\Delta/a$. We have written a program which solves (14) with matrix element (15) and a_n from (12),

$$a_n = \frac{-e^{i\xi\omega_0 n\Delta/\eta}qF_0(n\Delta)}{2Q_0} + \kappa D_n$$
$$+\frac{q^2}{4\pi Q_0P_0\omega_0}\sum_{k=0}^{2N+1}\Delta W_\perp(k\Delta)e^{i\xi\omega_0 k\Delta/\eta}D_{n-k}. \quad (16)$$

SIMULATIONS

The solution of Equations (1) and (2) using particle tracking is straightforward [5, 6, 7, 8]. Before considering a sinusiodaly drive BTF we consider forces of the form $F_0(\tau,\theta) = f(\tau)\cos(Q\theta)\exp(gQ)\delta_p(\theta)$ where δ_p is the periodic delta function, Q is the real part of the drive tune, g is the imaginary part of the drive tune and $f(\tau)$ is non-zero only over the full bunch length, $2\tau_b$. Suppose $D_f(\theta,\tau)$ is the dipole response to a kick $f(\tau)$ given at $\theta = 0$. Then the response from all the kicks is

$$D(\theta,\tau) = \sum_{m=-\infty}^{\infty} D_f(\theta - 2\pi m,\tau)\cos(2\pi mQ)\exp(2\pi gm). \quad (17)$$

To obtain a scalar we take the inner product

$$\hat{D}(\theta) = \int_{-\tau_b}^{\tau_b} d\tau f(\tau)D(\theta,\tau)$$
$$= \sum_{m=-\infty}^{\infty}\hat{D}_f(\theta - 2\pi m)\cos(2\pi mQ)\exp(2\pi gm)$$

For a pickup at a fixed location one obtains the time series $\hat{D}(0), \hat{D}(2\pi), \hat{D}(4\pi), \ldots$, these values may be obtained with a single simulation. As long as $\hat{D}_f(\theta)$ does not grow more quickly than $\exp(g\theta)$ we will have

$$\hat{D}(2\pi k) = [A_Q\cos(2\pi kQ) + B_Q\sin(2\pi kQ)]\exp(2\pi gk)$$

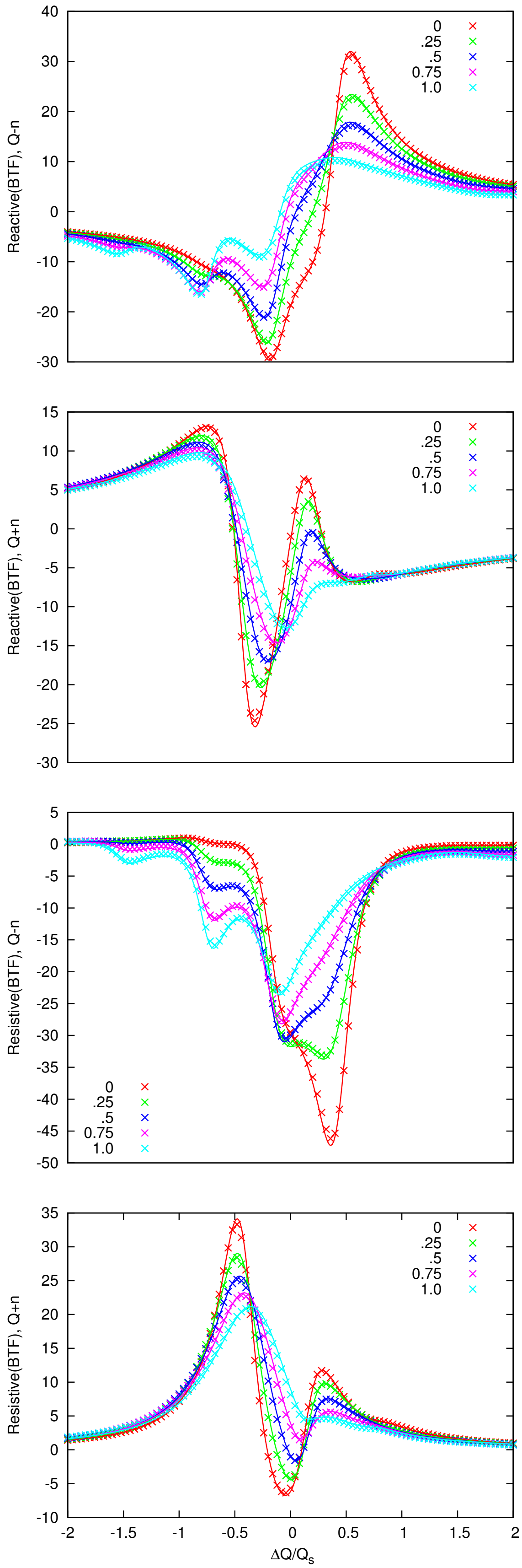

Figure 1: Simulated BTFs (solid lines) and those following from Eq (14) (crosses).

with

$$A_Q + jB_Q = \sum_{\ell=0}^{\infty} \hat{B}_f(2\pi\ell)\exp(2\pi\ell[jQ-g]) \quad (18)$$

Hence, the entire BTF can be obtained with one simulation. For a real beam transfer function the force at the pickup is sinusiodal. We assume the frequency is $n\omega_0 + \omega_1$ where n is an integer, $|\omega_1| \le \omega_0/2$. Then the driving force is given by

$$\begin{aligned} F_e(\theta,\tau) &= \delta_p(\theta)\cos[(n\omega_0+\omega_1)t], \\ &= \delta_p(\theta)\cos[(n\omega_0+\omega_1)(\tau+\theta/\omega_0)], \\ &\approx \delta_p(\theta)\cos(n\omega_0\tau)\cos(\omega_1 k(t)T_0) \\ &\quad -\delta_p(\theta)\sin(n\omega_0\tau)\sin(\omega_1 k(t)T_0) \quad (19) \end{aligned}$$

where $T_0 = 2\pi/\omega_0$ and $k(t) = \mathrm{nint}(t/T_0)$. In a real BTF only the response at the drive frequency is measured so, with a sinusiodal kick one needs to run two simulations. One with a kick proportional to $\sin(n\omega_0\tau)$ and another with kick $\cos(n\omega_0\tau)$. During each simulation one takes inner products of the response with each sinusoid every turn. One finds

$$BTF(\omega_1) = \sum_{m=0}^{\infty}(C_m + jS_m)e^{-(j\omega_1+\epsilon)mT_0}$$

with

$$C_m = \int_{-\tau_b}^{\tau_b} d\tau D_{cos}(2\pi m,\tau)e^{-jn\omega_0\tau},$$

where $D_{cos}(\theta,\tau)$ is the response to a cosine kick at $\theta = 0$, and similarly for S_m. In these equations we have used the electrical engineering convention with $j = -i$ and allowed for an exponentially growing drive $\propto \exp(\epsilon t)$. The driving terms in the Vlasov approach are given by $F_0(\tau) = \exp(\pm jn\omega_0\tau)$ for the $Q \mp n$ sidebands, respectively. Figure 1 shows upper and lower sideband BTFs with $n\omega_0 = \pm\pi/2\tau_b$, varying chromaticity and a step function wake of size comparable to what is needed for a mode coupling instability. The peak space charge tune shift is 4 times the synchrotron tune. For more extreme parameters the disagreement increases and we continue to look for errors.

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