REVISED VIEW OF BASIC FEL EQUATIONS AND A NONLINEAR VLASOV DESCRIPTION*

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Abstract

The standard one-dimensional FEL equations are revised to incorporate a new approach to the wiggle average, in a context of continuous frequency. It has a firmer mathematical foundation, and leads to an interesting reformulation in the time domain. The equations are interpreted as a nonlinear Vlasov system for a continuous distribution function of ponderomotive angle and energy spread. Numerical solutions are illustrated.

VLASOV THEORY OF THE FEL

A linearized version of the Vlasov or Vlasov-Klimontovich equation is accepted as the natural mathematical framework for an analytic discussion of the basic mechanism of the FEL, at distances up to and including the exponential growth region. At greater distances along the undulator a nonlinear saturation comes into play, and to understand that regime one usually resorts to a simulation based on a limited number of representative particles. I explore an alternative approach, based on the full nonlinear Vlasov equation. The general advantage of nonlinear Vlasov over particle simulations is in reduction of modeling noise, but there may also be opportunities for analytic studies of the nonlinear equation. This approach requires a representation of the phase space distribution as a smooth function. To account for granularity from the finite particle number in the startup of the SASE process, I take the initial distribution to be noisy but smooth in the mathematical sense, namely a truncated formal Fourier series of a random discrete particle distribution.

REVISED TREATMENT OF BASIC FEL EQUATIONS

I first review a different treatment of basic FEL equations in one degree of freedom. I work with equations for general continuous frequency ν, and perform the wiggle average at general ν, not just at integers. Moreover, the averaging is now in a form that is justified by the mathematical theory of averaging of differential equations. The new equations suggest a more radical step, namely to reverse integration orders so as to pass from the frequency domain to the time domain (more exactly the domain of the ponderomotive angle). The kernel in the new scheme, analogous to a wake potential, has compact support and intriguing properties.

I consider planar electron motion in a planar undulator, with z and x being longitudinal and transverse position coordinates. A general representation of the transverse electric field is [1]

$$E_x(z,t)=\int_{-\infty}^{\infty}\hat{E}_x(\nu,z)e^{i\nu k_1(z-ct)}d\nu\ , \tag{1}$$

assuming only that the field is smooth and decaying at infinity as a function of $\alpha=k_1(z-ct)$. Here $k_1=2\pi/\lambda_1$ is the wave number of the fundamental optical mode. A similar representation holds for the current density $J_x(z,t)$, in the sense of Fourier transforms of generalized functions if the source consists of point particles. A calculation invoking the wave equation and the slow variation approximation (dropping the second derivative), followed by the inverse FT with respect to α, yields the result

$$\frac{\partial\hat{E}_x(\nu,z)}{\partial z}=-\frac{Z_0}{2}\hat{J}_x(\nu,z)\ , \tag{2}$$

with SI units, Z_0 being the impedance of free space.

To derive $\hat{J}_x$ assume a volume density of electrons

$$n_e(z,t)=-\frac{e}{\Sigma_\perp}\sum_{j=1}^{N}\delta(z-z_j(t))\ , \tag{3}$$

where $\Sigma_\perp$ is an average transverse area of the beam. Then by applying the formula for velocity v_x in the undulator one finds

$$J_x(z,t)\approx\frac{1}{\gamma_0}K\cos\zeta\, n_e(z,t)\ ,\ \zeta=k_uz\ ,\ \gamma=\gamma_0(1+\eta)\ , \tag{4}$$

where $k_u=2\pi/\lambda_u$ is the undulator wave number and the approximation is $|\eta|\ll1$, used without comment henceforth. Now take the FT of J_x with respect to α, and for each j change the integration variable from α to u where $t=t_j(u)\ ,z_j(t_j(u))=u$. Next introduce the ponderomotive angle θ as a convenient phase space coordinate, namely

$$\theta=\alpha+\chi(\zeta)\ ,\quad \chi(\zeta)=\zeta+\xi\sin2\zeta\ ,\quad \xi=\frac{1}{2+4/K^2}\ , \tag{5}$$

and define

$$\theta_j(z)=\alpha_j(z)+\chi(\zeta)=k_1(z-ct_j(z))+\chi(\zeta)\ . \tag{6}$$

Then the FT of the current becomes

$$\hat{J}_x(\nu,z)=-\frac{ecNK}{\gamma_0\Sigma_\perp\lambda_1}w(\zeta,\nu)<e^{-i\nu\theta(z)}>\ , \tag{7}$$

$$w(\zeta,\nu)=\cos\zeta\ e^{i\nu\chi(\zeta)}\ , \tag{8}$$

$$<e^{-i\nu\theta(z)}>=\frac{1}{N}\sum_{j=1}^{N}e^{-i\nu\theta_j(z)}\ . \tag{9}$$

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Now to make contact with Vlasov theory imagine a smooth phase space distribution function $f(\theta,\eta,z)$. It and its projection are normalized to 1:

$$\int d\theta \int f(\theta,\eta,z)d\eta = 1\ , \quad \rho(\theta,z) = \int f(\theta,\eta,z)d\eta\ . \tag{10}$$

The mean value of $\exp(-i\nu\theta(z))$ over the discrete distribution will be identified with the mean value at z over the continuous distribution, which is the Fourier transform of the bunch form:

$$\frac{1}{2\pi} < e^{-i\nu\theta(z)} >= \frac{1}{2\pi}\int e^{-i\nu\theta}\rho(\theta,z)d\theta = \hat{\rho}(\nu,z)\ . \tag{11}$$

Returning to Eq.(2) and using (7) and (11) the field is calculated merely by integrating with respect to z. Then the equations of motion for η and θ can be derived in a standard way, so that they take the form

$$\frac{d\eta}{dz} = -\kappa \int w^*(\zeta,\nu)e^{i\nu\theta}d\nu \int_0^z w(\zeta',\nu)\hat{\rho}(\nu,z')dz'\ , \tag{12}$$

$$\frac{d\theta}{dz} = 2k_u\eta\ , \tag{13}$$

where

$$\kappa = \frac{k_1 Z_0 (eK)^2 N}{2mc\gamma_0^3\Sigma_\perp}\ . \tag{14}$$

Eq.(12) is stated for the case of zero initial field in the integration of (2). Now the differential equations (12) and (13) are to be integrated, using the Vlasov equation to update the distribution function, hence updating $\hat{\rho}(\nu,z')$ through (10) and an FT , at each integration step dz. The Vlasov equation is

$$\frac{\partial f}{\partial z} + \frac{d\theta}{dz}\frac{\partial f}{\partial\theta} + \frac{d\eta}{dz}\frac{\partial f}{\partial\eta} = 0\ . \tag{15}$$

The update is done by the method of local characteristics, "local" referring to the procedure of holding the electric force constant over an integration step in z. Thus

$$f(\theta,\eta,z+dz) = f(\theta - d\theta_{E(z)}, \eta - d\eta_{E(z)}, z)\ , \tag{16}$$

the subscripts indicating an evaluation of coordinate increments with the field having its values at z.

It is usual to make use of the average of the "wiggle factor" w over one undulator period at integer ν, which is given by a Bessel function expression $[JJ]_q/2$ at odd $\nu = 2q+1$, and is zero at even ν [2] The average is used to replace w in ν-neighborhoods of integers which are, unfortunately, not well defined. One can avoid the ambiguity and proceed more naturally as follows, for arbitrary continuous ν. Change the independent variable from z to $\zeta = k_u z$ and integrate in steps $\Delta\zeta = 2\pi p$, that is in steps of p undulator periods with p an integer (in practice a small integer, say 1 to 5, proves to be satisfactory). During a step, $\hat{\rho}(\nu,\zeta')$ is regarded as constant, so comes out of the integral in (12).

Note that local integrals of w are a phase factor times a function $g_p(\nu)$:

$$\int_{\zeta_n}^{\zeta_{n+1}} w(\zeta,\nu)d\zeta = e^{in\nu\Delta\zeta}\ \Delta\zeta\ g_p(\nu)\ ,$$

$$g_p(\nu) = \int_0^{2\pi p} \cos u\ \exp\big[i\nu(u - 2\xi\sin 2u)\big]du\ . \tag{17}$$

Then putting $\zeta_n = n\Delta\zeta$, the differential equation for $\zeta \in [\zeta_n, \zeta_{n+1}]$ has the form

$$\frac{d\eta}{d\zeta} = -\frac{\kappa}{k_u^2}\int w^*(\zeta,\nu)e^{i\nu\theta}d\nu\Big[\hat{\rho}(\nu,\zeta_n)\int_{\zeta_n}^{\zeta} w(\zeta',\nu)d\zeta' + \Delta\zeta\ g_p(\nu)\sum_{m=0}^{n-1}\hat{\rho}(\nu,\zeta_m)e^{im\nu\Delta\zeta}\Big] \tag{18}$$

So far the integrals of w have arisen naturally, without invoking the concept of averaging. Finally, averaging is introduced by replacing the r.h.s. of the differential equation (18) by its average over the interval $[\zeta_n, \zeta_{n+1}]$. This local averaging of the vector field of an ODE gives an approximation with rigorous error bounds attributed to Eckhaus [3]. With the definition

$$h_p(\nu) = \frac{1}{(\Delta\zeta)^2}\int_0^{\Delta\zeta} w^*(v,\nu)\Big[\int_0^v w(u,\nu)du\Big]dv\ , \tag{19}$$

the averaged equation on the interval $[\zeta_n, \zeta_{n+1}]$ is

$$\frac{d\eta}{d\zeta} = -\frac{\Delta\zeta\ \kappa}{k_u^2}\int e^{i\nu\theta}d\nu\Big[h_p(\nu)\hat{\rho}(\nu,\zeta_n) + e^{-in\nu\Delta\zeta}|g_p(\nu)|^2\sum_{m=0}^{n-1}\hat{\rho}(\nu,\zeta_m)e^{im\nu\Delta\zeta}\Big] \tag{20}$$

I take (13) and (20) as the basis for the numerical work of the following. The functions $|g_p(\nu)|^2$ and $h_p(\nu)$ (divided by their values at $\nu = 1$) are plotted in Fig.1 for $p = 3$. The peaks get narrower with increasing p.

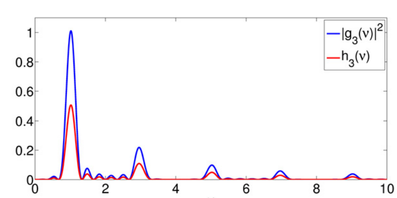

Figure 1: Functions appearing in Eq.20 (normalized)

A REFORMULATION IN THE θ DOMAIN

Introducing the definition (11) of $\hat{\rho}$ and reversing the order of integrations one can recast (20) in θ-space as

$$\frac{d\eta}{d\zeta} = -\frac{\kappa}{k_u^2}\int d\theta'\Big[T_p(\theta-\theta')\rho(\theta',\zeta_n) + \sum_{m=0}^{n-1} S_p(\theta-\theta'-\zeta_n+\zeta_m)\rho(\theta',\zeta_m)\Big]\ . \tag{21}$$

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The kernels S_p and T_p are

$$S_p(\theta) = p \int d\nu |g_p(\nu)|^2 e^{i\nu\theta} \; , \; T_p(\theta) = p \int d\nu h_p(\nu) e^{i\nu\theta} \; . \tag{22}$$

These functions have compact support (vanish outside a finite interval) as can be seen from the Paley-Wiener theorem [4]. Also $T_p(\theta) = S_p(\theta)$ for $\theta > 0$ and is zero for $\theta < 0$. The graph of S_3 is shown in Fig.2.

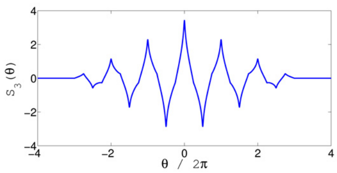

Figure 2: The kernel $S_p(\theta)$ for $p = 3$.

NUMERICAL VLASOV SOLUTIONS

To reduce the FEL equations to the simplest form, one assumes that the field and source are periodic in θ with period 2π, giving the so-called steady state case with identical motion in all buckets. More generally one can take a period $2\pi n_b$ of n_b buckets, and this can imitate the non-periodic case for large n_b. The following results are from a periodic code with arbitrary n_b, which represents f by its values on a grid, and implements the update (16) by bicubic interpolation to off-grid points. Parameters are for the LCLS: undulator parameter $K = 3.7$, Pierce parameter $\rho = 6 \cdot 10^{-4}$ (as defined in [1]). In terms of the normalized length $\bar{z} = 2\zeta\rho$ the gain length as given by linear theory is $1/\sqrt{3}$ ([1], §4.5.1).

I first take a single bucket, using a 400×400 grid in (θ, η)-space and $p = 1$ (integration step of one undulator period). The initial distribution is Gaussian in η with $\sigma = \rho$, times the truncated Fourier series of a random uniform distribution in θ of 50000 particles. There is no initial seed of the field. The calculation takes one minute on a single processor (2.3 GHz). Integration for 30 gain lengths gives the gain curve shown in Fig.3 (left). Fig.3 (right) shows the slope of the log of the power (blue) and the value $\sqrt{3}$ that this has in the linear theory (red). The linear prediction is good from about 5 to 12 gain lengths.

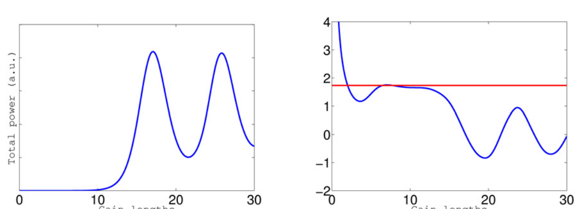

Figure 3: Left: power vs. number of gain lengths; Right: slope of log of power (blue) and its value in linear theory (red).

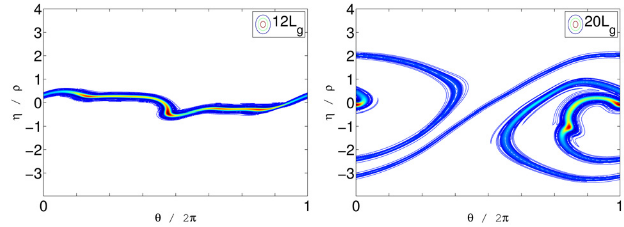

Figure 4: Phase space at 12 and 20 gain lengths.

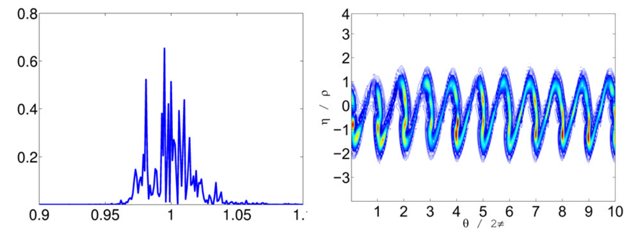

Figure 5: Field spectrum and phase space density, SASE model.

Fig.4 shows contour plots of phase space densities near the end of the exponential region and well into the saturation region.

To model the SASE mechanism one needs in principle a non-periodic θ dependence. To realize this numerically I take instead a long period of $n_b = 1000$ buckets, and also increase the Peirce parameter to $\rho = 0.01$, to decrease the computation time while still demonstrating the qualitative picture of SASE. I integrate to 8 gain lengths, at which point the coherence length $c\sigma_\tau$ is $4.8\lambda_1$ (by Eq.(4.155) in [1]), much less than the bunch length of $1000\lambda_1$; this should mean that the non-periodic case is well imitated. The frequency distribution in the field, $|\hat{E}_x(\nu)|^2$, appears near $\nu = 1$ as in Fig.5 (left). The expected number of spikes in a very crude estimate ([1], p.90) is $M \approx n_b\lambda_1/4c\sigma_\tau = 52$, not grossly different from what is seen. The phase space density in 10 adjacent buckets is shown in Fig.5 (right).

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