LINEAR ANALYSIS FOR SEVERAL 6-D IONIZATION COOLING LATTICES*

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Abstract

In muon accelerators, ionization cooling is the only viable option for reducing the muon beam emittance to values necessary for a muon collider. An ionization cooling lattice requires a large momentum acceptance, small beta functions, an extremely large dynamic aperture, and a modest amount of dispersion to have good performance. The latter values are a function of beam momentum. One step in understanding the lattice performance is to determine these quantities for a lattice cell being studied. This is modestly more complex than usual since the lattice is generally highly coupled. We first review the general method for computing these quantities for a general lattice. Then we look at several lattices of interest, compute these quantities, and relate their values and momentum dependencies to the performance of the lattices.

ANALYSIS

To try to understand the behavior of cooling lattices, one can used standard techniques for analyzing accelerator lattices. These systems are highly coupled, and they are non-symplectic due to the presence of cooling. The symplecticity is further complicated by the use of kinetic momenta in most tracking codes. Use of canonical momenta would be challenging due to the strong transverse vector potentials that arise from the use of solenoids, but would arise in any case due to the use of short cells and large beam sizes. Symplecticity can be further compromised by use of a non-symplectic integrator with step sizes larger than needed for machine precision and magnetic field approximations that are not divergence-free.

Fixed Energy

At fixed energies, we find the closed orbit through a single cooling cell, then look at the linear map about that closed orbit. We then find the eigenvalues and eigenvectors of the linear map to get the tunes and the beta functions. More specifically, if M is the linear map about the closed orbit, then we find a matrix A such that $MA = AR$, where R is in the form

$$\begin{bmatrix} \lambda_1\cos\mu_1 & \lambda_1\sin\mu_1 & 0 & 0 \\ -\lambda_1\sin\mu_1 & \lambda_1\cos\mu_1 & 0 & 0 \\ 0 & 0 & \lambda_1\cos\mu_2 & \lambda_1\sin\mu_2 \\ 0 & 0 & -\lambda_1\sin\mu_2 & \lambda_1\cos\mu_2 \end{bmatrix} \quad (1)$$

Since M is not necessarily symplectic, we allow $\lambda_k \neq 1$.

We find the lattice parameters as a function of energy. We will not study energies for which the lattice does not have stable orbits in the sense that R is of the form (1).

The matrix A is not uniquely defined. There are three different degrees of freedom. First, one can exchange the first two columns of A with the last two columns, which corresponds to swapping the eigenvalues. Second, one can change the sign of one of the first two columns of A, which corresponds to changing the sign of μ_1 (similarly for the last two columns and the sign of μ_2). Finally, one can multiply A on the right by a matrix of the form (1) (but with different parameters than R). We will define A sufficiently to accomplish three goals: to resolve the sign ambiguity in μ_k in R; to be able to compute the beta functions; and to ensure that the two sets of eigenvalues and the beta functions are continuous functions of the energy (so that the 1 and 2 indices don't swap arbitrarily).

For a given energy, we scale both $\boldsymbol{a}_{2k-1}$ and $\boldsymbol{a}_{2k}$ ($\boldsymbol{a}_j$, $1 \leqslant j \leqslant 4$, is the jth column of A) by the same value, and possibly change the sign of $\boldsymbol{a}_{2k}$, so that $\boldsymbol{a}_{2k-1}^T J \boldsymbol{a}_{2k} = 1$, where

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad (2)$$

and the order of coordinates is (x, p_x, y, p_y). If we change the sign of $\boldsymbol{a}_{2k}$, we also change the sign of μ_k. There is still an arbitrary rotation on the right side of A; we will insure that subsequent results are independent of that rotation.

Say we compute A_1 and A_2 for nearby energies. We would expect these matrices to be nearly the same, except for an arbitrary rotation on the right side of each one. If the columns of these matrices are $\boldsymbol{a}_{1;j}$ and $\boldsymbol{a}_{2;j}$, we compute the matrix S_{kl} with elements $s_{kl;ij} = \boldsymbol{a}_{2;2k-2+i}^T J \boldsymbol{a}_{1;2l-2+j}$. If $\det S_{11} > \det S_{12}$, then the columns of A_1 and A_2 refer to corresponding eigenvectors. If not, then the eigenvectors are in a different order in A_1 and A_2, and for one of the matrices the first pair of columns should be swapped with the second pair, and the eigenvalues should be swapped as well.

Each eigenvector pair describes a corresponding set of Courant-Snyder lattice functions. They relate the beam size and angular divergence to the area traced out by an ellipse. Since with coupling, oscillation is not in a fixed plane, the corresponding quantities must be independent of direction. They must also be independent of the arbitrary rotation on the right side of A. By analogy to the usual quantities, we can then define

$$\beta_k = a_{1,2k}^2 + a_{1,2k+1}^2 + a_{3,2k}^2 + a_{3,2k+1}^2 \quad (3)$$

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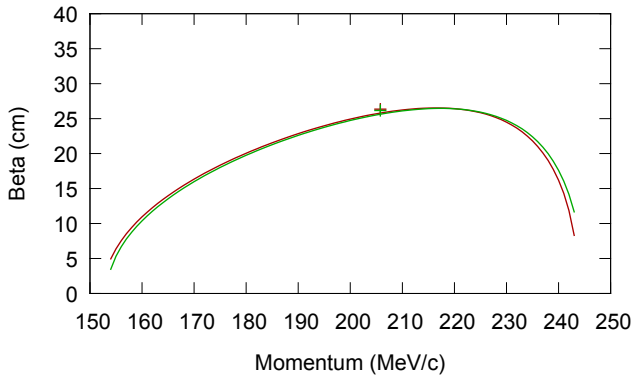

Figure 1: Beta functions vs. total momentum in stage 4 of the rectilinear FOFO (curves). The plus symbols are the beta functions and total momentum for the 6-D closed orbit.

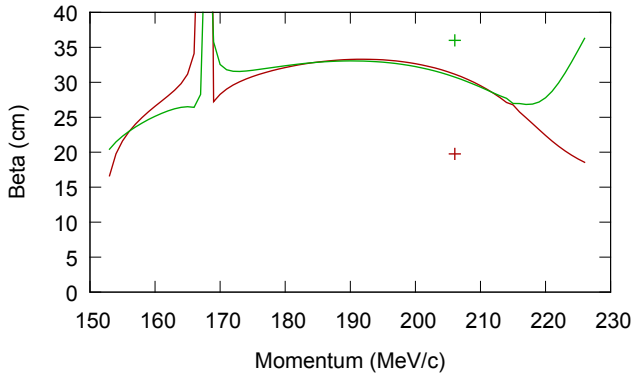

Figure 2: Like Fig. 1, but for the planar snake.

One could similarly define α_k and γ_k, though the usual relationships between these quantities do not necessarily hold since there are additional degrees of freedom due to coupling between the planes.

We calculate dynamic apertures as a function of energy by launching particles along the vector

$$\sqrt{2J_1}\boldsymbol{a}_1 + \sqrt{2J_2}\boldsymbol{a}_3 \tag{4}$$

and keeping track of the number of turns a particle survives as a function of J_1 and J_2.

6-D Fixed Point

We can also find the 6-D fixed point in the presence of acceleration and the average energy loss in the absorbers. All calculations are as for fixed energy but with 6-dimensional phase space vectors. Our additional variables are time and energy. The additional rows and columns of J may need a sign change or scaling factor depending on the exact implementation (blindly using energy and time, plus having momenta in eV/c, leads to a factor of $-c$ in J as used above). The eigenvalues will no longer have unit magnitude even for Maxwellian fields due to the presence of the absorber. Beta functions for the modes which are primarily transverse are computed according to (3).

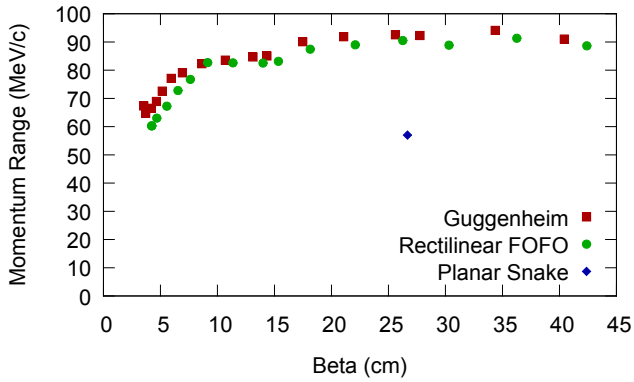

Figure 3: Momentum range vs. the geometric average of the beta functions at the 6-D fixed point.

One can calculate a "lossless" Q factor which arises solely from the average energy loss in the absorber and decays:

$$\frac{d\epsilon_6/\epsilon_6}{dN/N} = \frac{-2\ln|\lambda_1\lambda_2\lambda_3|(pc)\tau_\mu c}{L(m_\mu c^2)} \tag{5}$$

where ϵ_6 is the product of the emittances, N is the number of particles, λ_k are the eigenvalues of the one-cell map, p is the average total momentum for the closed orbit particle, τ_μ is the muon rest lifetime, m_μ is the muon rest mass, L is the cell length, and c is the speed of light. The Q value for the real lattice [1] is lower due to multiple scattering and energy straggling, which increase the emittances and cause particles to fall out of the dynamic aperture, dynamical losses of large amplitude particles, and emittance growth from mismatches and nonlinearities.

LATTICES

We study three cooling lattices. The first is a "Guggenheim" channel [2], where the beamline bends around in a circle while gradually moving perpendicular to the plane of the circle. A dipole field which is mostly in the vertical direction is generated by tilting the solenoid coils. In addition to generating bending, this creates dispersion (in position) at a wedge-shaped absorber. The solenoid field undergoes a single sinusoidal-like oscillation within the cell. The second is a rectilinear FOFO snake [3, 4], which has fields similar to the Guggenheim channel, but with smaller coil tilts and therefore a smaller dipole field, but where the beamline is straight. This requires vertical motion of the closed orbit to correct the bending from the dipole field. The third is a planar snake [5], which is a straight channel like the rectilinear FOFO. However, the solenoid field in one cell is opposite to the field in the next cell. In the presence of the dipole field generated from the tilts, this makes the period of this lattice two "cells" long. In addition, the direction of the tilts is chosen so that instead of dispersion in position at the absorber, there is dispersion in momentum. Thus a parallel-faced absorber can be used instead of a wedge.

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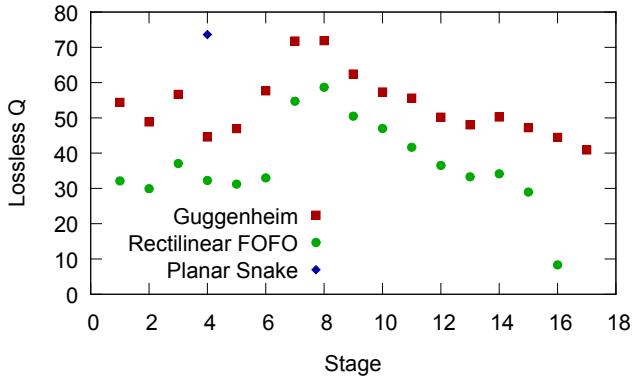

Figure 4: Lossless Q, vs. stage number. The planar snake has a beta function similar to stage 4 of either the Guggenheim or rectilinear FOFO.

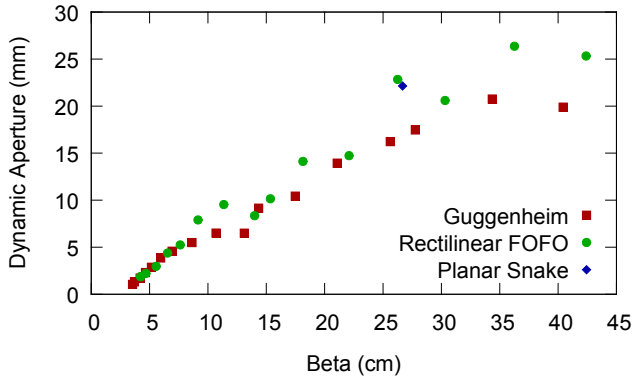

Figure 5: Dynamic aperture vs. the geometric average of the beta functions at the 6-D fixed point. Dynamic aperture is the average over the momentum passband of the smallest value of $J_x + J_y$ (normalized) beyond which there is at least one (J_x, J_y) pair which does not survive for 20 turns.

Calculations are performed using ICOOL [6] and computing linear maps via finite differences.

Figures 1 and 2 show the beta functions as a function of total momentum for a rectilinear FOFO (the Guggenheim is similar) and the planar snake. The rectilinear FOFO and Guggenheim operate in the passband between π and 2π phase advance. The planar snake operates similarly, except that the period is two cells, so what is shown in Fig. 2 is for a phase advance from 2π to 4π. There is thus a linear resonance (3π) near 168 MeV/c, which despite its weakness appears to be enough to truncate the operating passband. One also observes a strong change in the beta functions at the high energy end (there is a corresponding change in the closed orbit) arising from the approach to an integer tune. There are also indications of a coupling resonance near 213 MeV/c.

Figures 1 and 2 also show the beta functions at the 6-D fixed point. For the rectilinear FOFO and Guggenheim, the beta functions are close to what is found in the fixed energy calculation. For the planar snake, however, the beta functions are very different. At the fixed point, the two betatron tunes and the synchrotron tunes have the same fractional part (near 0.25), and one of the transverse eigenvectors and the longitudinal eigenvector have ellipse projections that have nearly equal areas in the transverse and longitudinal directions. Thus the planar snake appears to be operating on a synchro-betatron resonance, similarly to what is described in [7].

Energy acceptance of a lattice is important both to achieve good longitudinal cooling and acceptance as well as to limit losses from energy straggling. One can look at the momentum acceptance that can be achieved for a given beta function at the absorber. The result is shown in Fig. 3. The planar snake has a significantly reduced momentum acceptance compared to the other lattices. The rectilinear FOFO (at least as designed) does not perform quite as well as the Guggenheim for smaller beta functions.

Figure 4 shows the lossless Q for the lattices. Note that the lossless Q is significantly lower for the rectilinear FOFO than for the other lattices. Since this is mainly related to the energy lost or gained in each cell, and the RF gradients in the rectilinear FOFO are similar or larger than those used in the Guggenheim or planar snake, one would expect that the rectilinear FOFO could have improved performance, probably via some increase in the wedge thickness and subsequent tuning. This may explain the lower Q value found for the rectilinear FOFO (maximum of 9.5 [4]) compared to the Guggenheim (maximum of 11 [2]).

Figure 5 shows the dynamic aperture plotted against the beta function for the Guggenheim and rectilinear FOFO lattices. Note that the dynamic aperture is not proportional to the beta function for small beta (it is worse), which ultimately prevents achieving smaller emittances.

REFERENCES

[1] R. Palmer *et al.*, "Ionization cooling ring for muons," Phys. Rev. ST Accel. Beams **8**, 061003 (2005).

[2] D. Stratakis, R. C. Fernow, J. S. Berg, and R. B. Palmer, "Tapered channel for six-dimensional muon cooling toward micron-scale emittances," Phys. Rev. ST Accel. Beams **16**, 091001 (2013).

[3] V. Balbekov, "R_FOFO snake channel for 6D muon cooling," MAP-doc-4365 (2013). `http://map-docdb.fnal.gov/`

[4] D. Stratakis, J. S. Berg, R. B. Palmer, and V. I. Balbekov "A High-Performance Rectilinear FOFO Channel for Muon Cooling," to appear in the proceedings of the 2013 North American Particle Accelerator Conference (2013).

[5] R. B. Palmer, J. S. Berg, R. C. Fernow, and D. Stratakis, "A Planar Snake Muon Ionization Cooling Lattice," to appear in the proceedings of the 2013 North American Particle Accelerator Conference (2013).

[6] F. C. Fernow, "Recent Developments on the Muon-Facility Design-Code ICOOL," in *Proceedings of 2005 Particle Accelerator Conference, Knoxville, Tennessee* (IEEE, 2005) 2651.

[7] J. S. Berg, "Ionization cooling in all phase space planes with various absorber shapes, including parallel-faced absorbers," Nucl. Instrum. Methods A **569** (2006) 677.