Distribution of Eigenmodes in a Superconducting Stadium Billiard and " Quantum Chaos "

H.-D. Gräf[1], H. Lengeler[2], C. Rangacharyulu[3], A. Richter[1], T. Rietdorf[1] and P. Schardt[1]

[1]Institut für Kernphysik, Technische Hochschule Darmstadt, Germany
[2]CERN, AT Division, Switzerland
[3]Department of Physics, University of Saskatchewan, Saskatoon, Canada

I. Introduction

The transition of a classical system from integrable to non-integrable behaviour reflects itself as a change of the quantum mechanical eigenvalue spectrum from a regular (Poisson) distribution to that of Gaussian Orthogonal Ensemble (Wigner) type. A two dimensional radio frequency cavity provides an opportunity to test in experiment whether the assumption that a stadium or Sinai billiard shows chaotic behaviour holds true.
For this system, the correspondence between classical and quantum mechanics can be written as:

Classical:	Quantum mechanics:
$\nabla^2 \vec{E} = - k^2 \vec{E}$ (Maxwell)	$\nabla^2 \Psi = - k^2 \Psi$ (Schrödinger)
$k = \frac{2\pi\nu}{c}$	$k = \left[\frac{2mE}{\hbar^2} \right]^{1/2}$

A two dimensional cavity has been constructed by limiting one side of the cavity to

$$d < \frac{c}{2\nu_{max}} ,$$

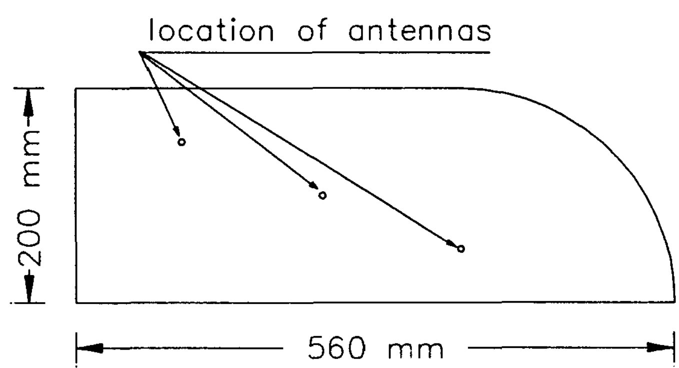
where ν_{max} is the maximum frequency of interest.
With $d = 8mm$ this system satisfies the two-dimensional criterion for frequencies below 18.75 GHz. With this cavity, the spectrum of eigenmodes has been measured at room temperature and in the superconducting state at 2K. First results of statistical analyses are presented.

*) Work supported by the Bundesministerium für Forschung und Technologie under contract number 06 DA 1847 and by the EURO FEL Network under contract number SC1 -0471-C.

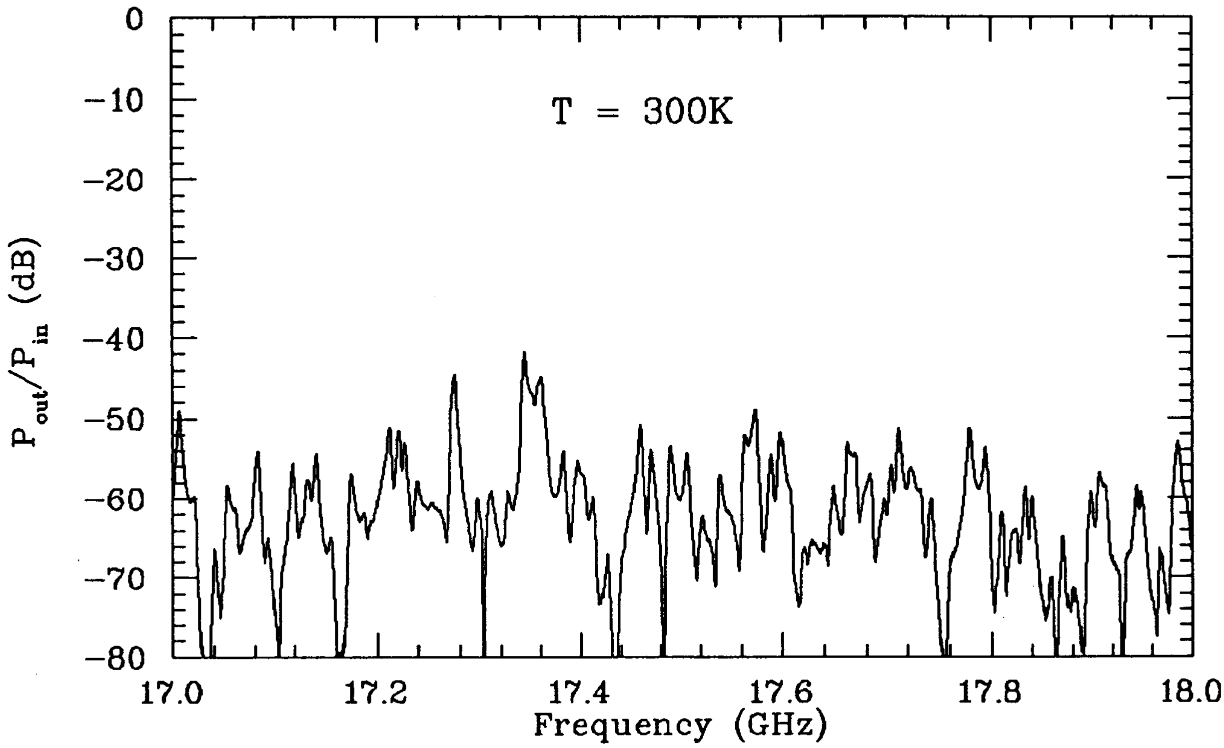
II. Measurements

Stimulated by the work of [1] a cavity shaped like a quarter of a stadium billiard was built out of niobium (fig. 1) and put into one of the cryostats of the superconducting Darmstadt electron linear accelerator (S-DALINAC) on which is being reported elsewhere in these Proceedings [2]. Only a quarter was choosen to avoid splitting of degenerate modes due to inaccuracies of the shape. We installed three antennas coupling to the z-component of the electrical field. They are located in small holes of 3 mm diameter not penetrating into the cavity to keep their influence on the field distributions of the modes negligibly small. By measuring the transmitted power (to get rid of influences of cables and connectors) we have obtained three independent spectra in order to detect modes with a node at the location of one antenna.

Fig. 1 Shape of the super-conducting cavity with antenna locations



A single spectrum consists of $1.8*10^6$ data points in the range from 0 to 18.75 GHz. The resolution of the measurement has been 10 kHz, whereas the largest width of modes detected was smaller than 30 kHz, which is also the smallest detectable spacing between two modes. In contrast the smallest observed spacing was 300 kHz, so we can be certain not to have missed modes (which unavoidably is true in the measurements with a low quality factor at room temperature). With the redundant information of the three spectra we counted 1253 eigenvalues at 2K compared to 993 eigenvalues at room temperature.
To illustrate the high resolution, a part of the spectrum is shown in fig. 2, where the upper spectrum was taken at room temperature and the lower one at 2 K. One also should notice the increase of the signal to noise ratio by more than 20dB.
The transition of the cavity into the superconducting state is illustrated in fig. 3 by a measurement of the quality factor Q during cooldown for one mode at 10.17450 GHz. The quality factors obtained are in the range of 10^5- 10^7.



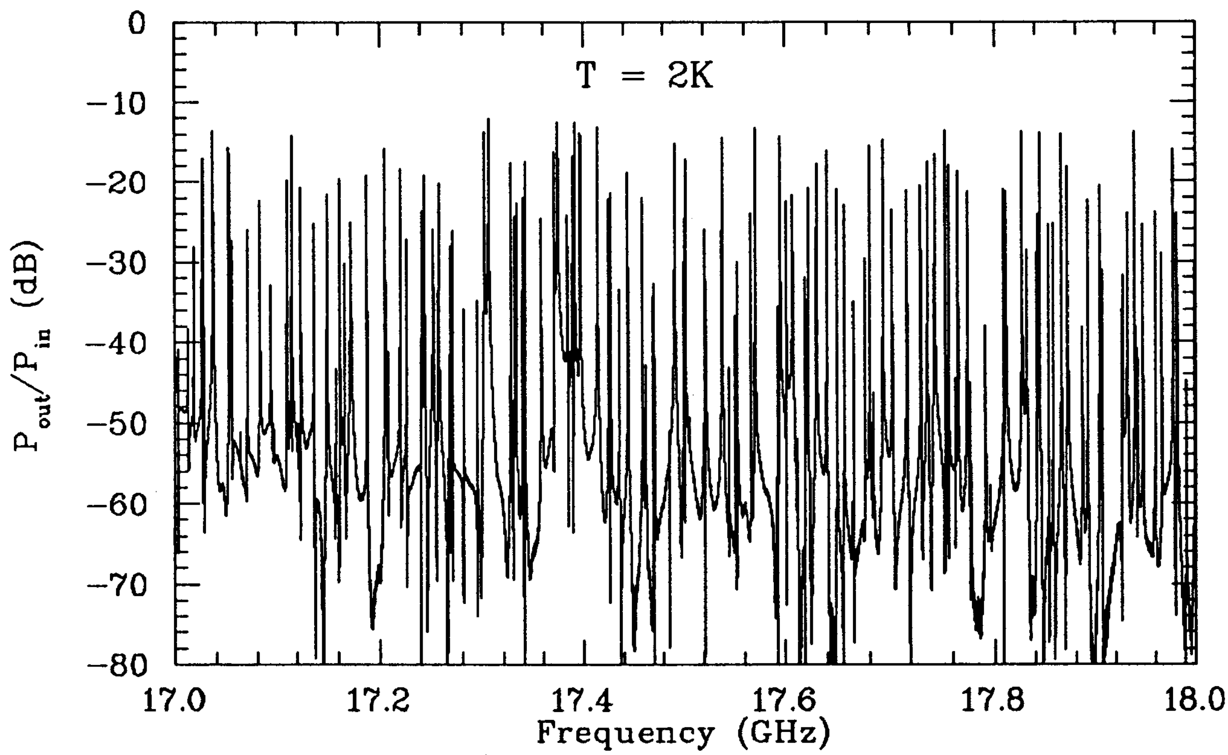


Fig. 2 Part of the spectrum from 17 to 18 GHz at room temperature (upper figure) and in the superconducting state at 2K (lower figure). The incident power was at a level of 10dBm.

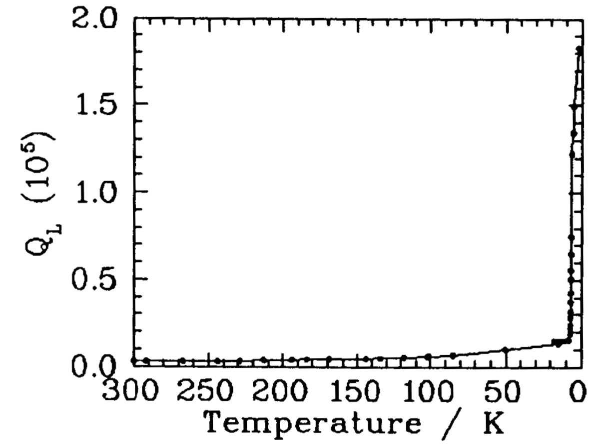


Fig. 3 Transition into the superconducting state during cooldown

III. Analysis of the Eigenmode Spectrum

In the first step of the analysis the nearest neighbour distribution shown in fig. 4 was investigated. One can fit the parameters of the general distribution [3] of the spacings s between the modes in units of the mean level spacing D

$$P(s) = C s^{\omega} \exp(-\alpha s^{\omega+1}), \quad s = (E_{i+1} - E_i)/D,$$

where ω is the repulsion parameter. The quantities C and α are found from the condition that both the average of s and the area under the curve must be equal to unity. The Poisson (regular) case is obtained for ω=0, the Wigner (chaotic) case for ω=1 :

- regular: $P(s) = \exp(-s)$

- chaotic: $P(s) = \frac{\pi s}{2} \exp(-\frac{\pi}{4} s^2)$,

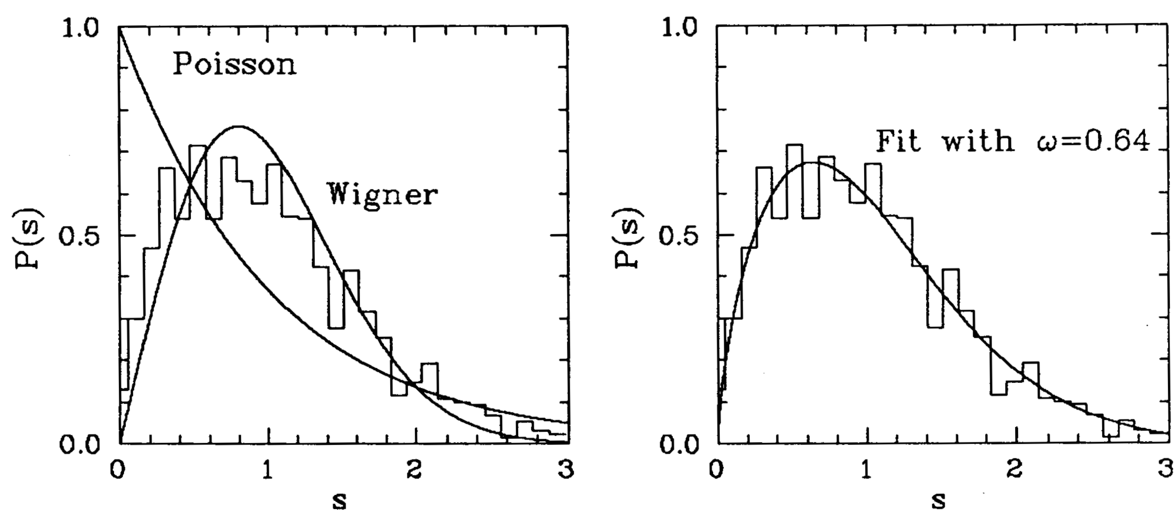


Fig. 4 Nearest neighbour distribution of 1253 eigenvalues. Left: regular (ω=0) and chaotic (ω=1) cases, right: parameter fit to the general P(s)-distribution

The fit results in a value of ω=0.64 (right part of fig. 4), i.e. the distribution of the spacings of eigenmodes is intermittend between regular and chaotic behaviour.

Also the Δ_3 statistic given by Dyson and Mehta [4]has been tested. It is defined as the minimal deviation of the cumulative spectrum N(E) from a straight line A·E+B:

$$\Delta_3(L) = \frac{1}{L} \left\langle \min \int_{L-E/2}^{L+E/2} [N(E) - A*E - B]^2 \, dE \right\rangle,$$

with L and E in units of D. The limiting cases are

-regular: $\Delta_3(L) = \frac{L}{15}$

-chaotic: $\Delta_3(L) = \frac{1}{\pi^2} \ln L - 0.007$.

The result (shown in fig. 5) is close to the Wigner distribution, but exhibits a deviation for values of L greater than 20. This shows the existence of some levels not belonging to the GOE ensemble.

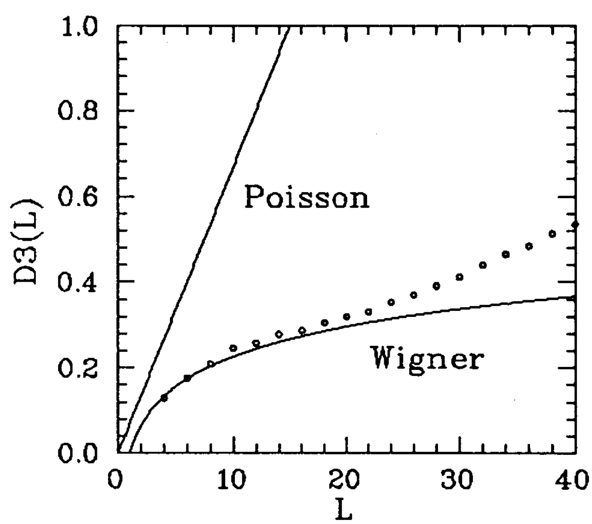


Fig. 5 Δ_3- Statistic of 1253 eigenmodes with the limiting cases for Poisson and Wigner statistics

This can also be seen in the F-statistics test formulated by Dyson [5]. A Wigner ensemble would show a Gaussian distribution with an expectation value for F_i at 27.24 and a width of 1.89. In contrast we observe a skew-symmetric distribution as shown in fig. 6. Its shape clearly provides a hint to the existence of spurious levels.

All tests have been applied in three frequency regions : 0 - 10 GHz, 10 - 15 GHz and 15 - 18.75 GHz. There was no significant dependence on frequency in any case. This confirms, that we are not missing modes at higher level density and do not observe any influences of the antennas in contrast to measurements at room temperature [6].

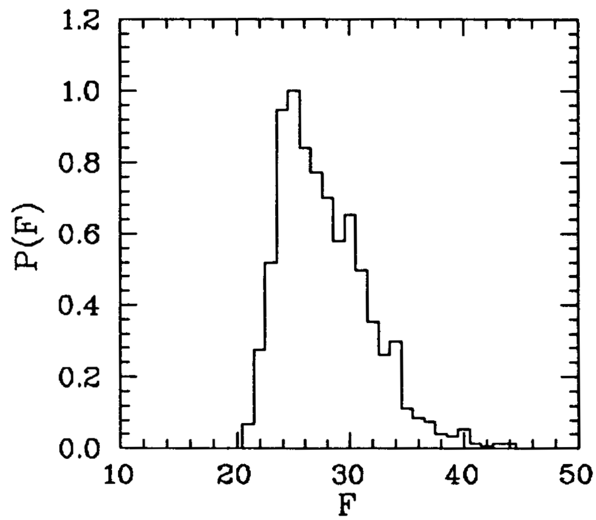


Fig. 6 Spurious levels cause greater F-values than expected, so the distribution is not gaussian at all

The F-statistics test is a powerful tool to find out the modes, which cause the deviations from a Gaussian. We extracted about 100 modes to get the F-distribution into Gaussian shape with a width of $\sigma = 1.4$. The edited spectrum has become Wigner distributed with a repulsion parameter $\omega = 0.98$, while the extracted modes show regular behaviour.

A Fourier transform into the space of periodic orbit length l as proposed in [1]

$$c(l) = \sum_n \exp(i k_n l) \qquad k_n = \frac{2\pi\nu_n}{c},$$

where the sum is over all eigenmodes with frequency ν_n, exhibits the regular nature of the 100 extracted modes. The resonances are at multiples of 0.4m, which is exactly the length of the shortest "bouncing-ball" orbit in the cavity (fig.7).

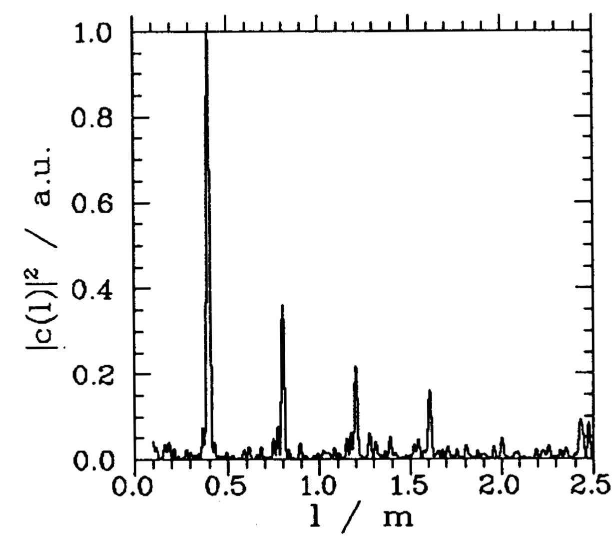


Fig. 7 Fourier transform of the spectrum made out of the extracted modes

We continue presently our investigations of the spectrum of the eigenmodes.

IV. Acknowledgement

We feel very obliged to the technical staff at the CERN- workshop for the exact manufacturing and careful preparation of the cavity.

V. References

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