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Optimization of the Cooling Power Distribution in a Superconducting Linac[$]

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The benefits of setting the resonators in a superconducting heavy-ion linac to a certain optimum distribution of cooling power have been evaluated in terms of the total acceleration such a distribution may produce, compared to a distribution in which each resonator dissipates power equally. The optimum power distribution can be expressed in closed form in certain simplified cases, but the general case is solved by equalizing the "marginal power cost" of the resonators by iteration in a computer simulation. For the Stony Brook linac an additional possible acceleration of several percent is thus predicted for typical beams.

1 Introduction

The practical operating fields of lead-coated copper cavities in superconducting heavy-ion linacs like those at Stony Brook, Seattle and INFN-Legnaro are more likely to be set by the tolerable heat dissipation than by some hard limit like thermal breakdown. It follows that the total energy gain achievable in these machines is determined primarily by the net available installed refrigeration. The traditional practice is simply to distribute this available cooling power *equally* among all of the active resonators by appropriate adjustments of their operating fields. This setup procedure, while very convenient, clearly does not necessarily assure the highest possible acceleration, but before now there was no published information on how to do the field setting better or on what additional acceleration could thereby be gained.

The present work provides a new systematic method for finding the *optimum power distribution* in such a superconducting linac, that is, the distribution of resonator fields and associated power dissipations that maximizes the total acceleration for a fixed total power. The method uses an iterative computer procedure to equalize the *marginal power cost* of running each resonator, a condition that is shown to be equivalent to a solution of the problem.

2 Mathematical preliminaries

2.1 Acceleration and Power

The acceleration produced by the nth resonator in a heavy-ion linac is given by

$$W_n = q\mathcal{E}_n L_n T_n(\beta_n)\cos\phi \tag{1}$$

where q is the ion charge, $\mathcal{E}_n$ is the accelerating field in MV/m, L_n is the nominal resonator length in meters, $T_n(\beta_n)$ is the transit-time factor when a beam enters the resonator with velocity β_n, and ϕ is the phase of the resonator field relative to the mean arrival time of the beam bunch. (We ignore here any second order effects in $T_n(\beta_n)$ caused by changes in velocity within one resonator.) It is customary to use a normalized form for the transit-time factor in which $T(\beta_{\mathrm{opt}}) = 1$ and to take L_n to be the inside diameter of the cavity along the beam direction. The particle velocity is related to its energy E in MeV and its mass A in amu by $\beta = v/c = 0.0463\sqrt{E/A}$.

The rf power P_n dissipated by the nth resonator is given by

$$P_n = \frac{\omega U_n}{Q_n} = \frac{\omega \kappa_n \mathcal{E}_n^2}{Q_n} \tag{2}$$

where U_n is the energy content, Q_n is the quality factor, $\omega = 2\pi f$ is the angular frequency and κ_n is the cavity constant that relates U_n to $\mathcal{E}_n^2$.

The total acceleration W and total power dissipation P of a linac are obtained by summing the above expressions over all of its resonators. Quantities common to all resonators, or to a group of resonators, may be taken out of the summations, simplifying the expressions. In the present work q, ω, and ϕ are considered to be common to all resonators while L_n and κ_n are allowed to vary. (For the Stony Brook linac $f = 150.4$ MHz, ϕ is normally $-15°$, and there are two types of resonators optimized for different beam velocities.)

The task of maximizing W for a given total P is complicated by the fact that the acceleration of a particular resonator depends on the velocity of the beam entering it through the factor $T(\beta)$. This implies a coupling between the various resonators in a linac. Changing the field level of a given resonator causes both a direct change in the total linac acceleration and various indirect changes by virtue of the changing transit-time factors of all of the following resonators. A second complication is that, except perhaps at very low fields, Q is not a constant in superconducting resonators but rather a monotonically decreasing function of $\mathcal{E}$.

2.2 Approximating $Q(\mathcal{E})$ and $T(\beta)$

The solution of the general optimization problem by numerical simulation requires a reasonable mathematical description of the $Q(\mathcal{E})$ and $T(\beta)$ functions for each resonator or type of resonator, respectively. Considering first $Q(\mathcal{E})$, one could attempt to develop an appropriate form for this from assumptions about various possible loss mechanisms, but for the present work it was more convenient just to use simple polynomial approximations. A typical approximation has the form

$$Q(\mathcal{E}) = a_0 - a_2\mathcal{E}^2 + a_3\mathcal{E}^3 + a_4\mathcal{E}^4 - a_{12}\mathcal{E}^{12},$$

in which a_0, a_2 and a_{12} are positive numbers, a_{12} is very small, and in many cases, a_3 and/or a_4 are zero. The absence of a linear term assures that $Q(\mathcal{E})$ flattens out at low field as expected, while the twelfth degree term produces the usual sharp drop off in Q at high fields. Coefficients were derived from measured Q curve points by standard least-squares fitting methods.

The accelerating portion of the Stony Brook linac consists of 16 low-beta ($\beta_{\rm opt} = 0.068$) quarter-wave resonators, with two equal beam gaps, followed by 24 high-beta ($\beta_{\rm opt} = 0.10$) split-loop resonators, with one large center gap and two smaller outer gaps. The transit-time factor curves for the two resonator types are quite well reproduced by the following two functions

$$T_{\rm QWR}(\beta) = T_1(\frac{\beta}{\beta_1})\sin\frac{\beta_1}{\beta}\sin\frac{\beta_s}{\beta}, \text{and} \quad (3)$$

$$T_{\rm SLR}(\beta) = T_1(\frac{\beta}{\beta_1})\sin\frac{\beta_1}{\beta} - T_2(\frac{\beta}{\beta_2})\sin\frac{\beta_2}{\beta}\cos\frac{\beta_s}{\beta}\pi \quad (4)$$

containing the following constants:

	T_1	T_2	β_1	β_2	β_s
QWR:	1.11		0.050		0.0938
SLR:	0.583	0.600	0.117	0.0707	0.0938

The derivatives $\frac{dT}{d\beta}$ and $\frac{dQ}{d\mathcal{E}}$ needed for the iterative procedure discussed in Section 4 are evaluated from explicit expressions derived from the above [1].

3 Marginal Power Cost

3.1 The derivative condition

The first step in the general solution of the optimization problem is to derive a key identity. We begin by defining two N-dimensional vectors to represent the distributions of cooling power and acceleration in a series of N resonators,

$$\mathbf{P} \equiv (P_1, P_2, \cdots, P_N)$$

$$\mathbf{W} \equiv (W_1, W_2, \cdots, W_N)$$

The power distribution, $\mathbf{P}$, is a one-to-one function of the acceleration distribution $\mathbf{W}$; that is, any particular distribution of accelerations among the resonators will determine a unique power distribution, and vice versa. Two more functions, which describe the total acceleration and total power, can be written as summations or, more elegantly, in terms of a dot product with the N-dimensional unit vector $\mathbf{I}_N$:

$$W(\mathbf{W}) \equiv \sum_n W_n = \mathbf{I}_N \cdot \mathbf{W}$$

$$P(\mathbf{W}) \equiv \sum_n P_n = \mathbf{I}_N \cdot \mathbf{P}(\mathbf{W})$$

In the latter equation, P_n is the nth component of the vector function $\mathbf{P}(\mathbf{W})$.

The problem is now to maximize $W(\mathbf{W})$ while keeping $P(\mathbf{W})$ constant. In other words, W must be maximized, given the condition that the function $g(\mathbf{W})$, defined as

$$g(\mathbf{W}) \equiv P(\mathbf{W}) - P_{total},$$

is equal to zero, for a fixed total power, P_{total}. The well-known theorem of Lagrange multipliers allows us to say that there exists some constant number λ such that

$$\lambda \nabla W = \nabla g = \nabla P$$

for the distribution $\mathbf{W}$ we are looking for. Now the N-dimensional gradient of $W(\mathbf{W})$ is simply the constant vector $\mathbf{I}_N$, thus we end up with the following key result:

$$\nabla P = \lambda \mathbf{I}_N \tag{5}$$

Equation 5 means that the components of $\nabla P(\mathbf{W})$ must all be equal to the same constant λ and hence all equal to each other.

3.2 Defining MPC

At this point in the discussion we introduce the concept of *marginal power cost* or MPC. We define the marginal power cost M_n of operating resonator n at accelerating field $\mathcal{E}_n$ as the partial derivative $\frac{\partial P}{\partial W_n}$ of the total linac power P with respect to the acceleration W_n produced by resonator n. As M_n is simply the nth component of ∇P in Equation 5 above it follows that:

> *The optimum power distribution has an equal marginal power cost for each resonator.*

The marginal power cost for resonator n can be expanded as follows:

$$M_n \equiv \frac{\partial P}{\partial W_n} = \frac{\partial}{\partial W_n} \sum_m P_m = \sum_m \frac{\partial P_m}{\partial W_n} \tag{6}$$

where P_m is the power dissipated in resonator m. The summation at the right contains one *direct term* with $m = n$ and up to $N-1$ *indirect terms* with $m > n$. The direct term relates to the change in power dissipation of resonator n due to a change in its own acceleration, while the indirect terms relate to power changes in downstream resonators as their transit-time factors change. There are no non-zero indirect terms with $m < n$ because resonator n cannot affect earlier resonators in the linac.

3.3 Evaluating the $m = n$ term

It is convenient to call the direct or $m = n$ term of Equation 6 μ_n:

$$\mu_n \equiv \frac{\partial P_n}{\partial W_n}.$$

To find an expression for μ_n, Equation 2 must be written in terms of W_n using Equation 1. This yields

$$P_n = \frac{\omega \kappa_n W_n^2}{Q_n q^2 L_n^2 T_n^2(\beta_n) \cos^2 \phi}.$$

The only variables to consider in evaluating μ_n are W_n and Q_n (which changes as $\mathcal{E}_n$ changes with W_n). The transit-time factor is *not* involved because β_n is assumed to be the beam velocity just *before* resonator n. Taking the partial derivative of P_n with respect to W_n, and expressing the result in terms of the accelerating field,

$$\mu_n = \frac{\omega \kappa_n \mathcal{E}_n}{Q_n q L_n T_n(\beta_n) \cos \phi} \left(2 - \frac{\mathcal{E}_n}{Q_n} \frac{dQ_n}{d\mathcal{E}_n}\right), \tag{7}$$

which can also be written

$$\mu_n = \frac{P_n}{W_n} \left(2 - \frac{\mathcal{E}_n}{Q_n} \frac{dQ_n}{d\mathcal{E}_n}\right). \tag{8}$$

It is evident from Equation 8 that the direct marginal power cost μ_n is always positive and always increasing with increasing power P_n *at least* as fast as $P^{\frac{1}{2}}$ and possibly much faster. This follows from the fact that Q is invariably either constant or falling with increasing accelerating field (making $\frac{dQ_n}{d\mathcal{E}_n} \leq 0$) and the related fact that the acceleration W_n never increases any faster than $P_n^{\frac{1}{2}}$.

3.4 An exactly-solvable case

Before going further it is instructive to imagine a hypothetical very simplified case in which both

Q_n and T_n are assumed to be fixed constants for each resonator rather than functions of $\mathcal{E}$ and β respectively. With constant transit-time factors only the direct term μ_n of the marginal power cost will be non-zero. Then with Q_n also constant Equation 8 reduces simply to

$$\mu_n = 2\frac{P_n}{W_n}.$$

Noting that μ_n has the same value for all n resonators at the optimum distribution, and that $P = \sum P_n$, the immediate result is an explicit expression for the optimum P_n's

$$P_n^{\rm opt} = \frac{L_n^2 T_n^2 Q_n}{\xi^2 \kappa_n}, \tag{9}$$

where ξ^2 is a normalization constant given by

$$\xi^2 = \sum_n \frac{L_n^2 T_n^2 Q_n}{\kappa_n}.$$

Equation 9 clearly illustrates the intuitive rule-of-thumb that "resonators that accelerate better should be given proportionally more power." Better acceleration is achieved in this case when L is greater, T is closer to unity, Q is higher, and/or κ is smaller. Calculated optimum power distributions were tested for various hypothetical sets of Q and T values in a QBasic simulation. The improvement in acceleration over the even power distribution ($P_n = P/N$) varied from 1% to more than 10% depending on how wide a Q distribution was assumed.

3.5 Evaluating the $m > n$ terms

We now consider the indirect MPC terms related to the changing transit-time factors in resonators downstream of resonator n. Careful application of the chain rule gives the first tool to the solution of this problem,

$$\frac{\partial P_m}{\partial W_n} = \frac{dP_m}{d\mathcal{E}_m}\frac{\partial \mathcal{E}_m}{\partial W_n}. \tag{10}$$

An expression for $\frac{dP_m}{d\mathcal{E}_m}$ is easily found by differentiating Equation 2, which yields

$$\frac{dP_m}{d\mathcal{E}_m} = \frac{\omega\kappa_m\mathcal{E}_m}{Q_m}\left(2 - \frac{\mathcal{E}_m}{Q_m}\frac{dQ_m}{d\mathcal{E}_m}\right) \tag{11}$$

Now the problem is reduced to the evaluation of the more complicated quantity $\frac{\partial \mathcal{E}_m}{\partial W_n}$. Solving Equation 1 for the accelerating field,

$$\mathcal{E}_m = \frac{W_m}{qL_m T_m(\beta_m)\cos\phi}.$$

For $n < m$, a small change in W_n will produce a change in the transit time factor at resonator m, but every other quantity in the relation above remains constant. With this in mind,

$$\frac{\partial \mathcal{E}_m}{\partial W_n} = -\frac{W_m}{qL_m\cos\phi}\frac{1}{T_m^2(\beta_m)}\frac{dT_m}{d\beta_m}\frac{\partial \beta_m}{\partial W_n}. \tag{12}$$

Having reduced the problem further, we now need to know how much the velocity at resonator m is affected by a small change in the acceleration at resonator n. In terms of the linac injection energy E_0 and the components of the acceleration vector $\mathbf{W}$, the velocity β_m can be expressed as

$$\beta_m = 0.0463\sqrt{\frac{E_0 + \sum_{i=1}^{m-1} W_i}{A}}$$

A small change in W_n changes only the one term in the summation for which $i = n$. Since every other term remains constant, the partial derivative is simply:

$$\frac{\partial \beta_m}{\partial W_n} = \frac{0.0463}{2A\sqrt{\frac{E_0+\sum_{i=1}^{m-1} W_i}{A}}} = \frac{(0.0463)^2}{2A\beta_m}. \tag{13}$$

We are now in a position to derive an expression for $\frac{\partial P_m}{\partial W_n}$ by combining Equations 10, 11, 12 and 13. The end result is

$$\frac{\partial P_m}{\partial W_n} = \frac{(0.0463)^2\omega\kappa_m\mathcal{E}_m^2}{2A\beta_m Q_m T_m(\beta_m)}\frac{dT_m}{d\beta_m}\left(\frac{\mathcal{E}_m}{Q_m}\frac{dQ_m}{d\mathcal{E}_m} - 2\right), \tag{14}$$

or in terms of P_m,

$$\frac{\partial P_m}{\partial W_n} = \frac{(0.0463)^2 P_m}{2A\beta_m T_m(\beta_m)}\frac{dT_m}{d\beta_m}\left(\frac{\mathcal{E}_m}{Q_m}\frac{dQ_m}{d\mathcal{E}_m} - 2\right). \tag{15}$$

These equations, together with Equation 7 for the direct term μ_n and the explicit derivatives of the $Q(\mathcal{E})$ and $T(\beta)$ functions discussed in Section 3.4 make possible a complete numerical calculation of the marginal power cost M_n of resonator n at any particular field $\mathcal{E}$. It should be noted that the contribution of these indirect terms to M_n could be either positive or negative depending on the sign of $\frac{dT_m}{d\beta_m}$.

4 Solution by iteration

The fact that each resonator must have the same marginal power cost at the optimum power distribution is useful, but it does not provide an explicit overall solution except in the simplified case described in Section 3.4. For this reason, a QBasic program was written to find the optimum power distribution for the Stony Brook linac numerically by an iterative procedure that repeatedly shifts power away from resonators with relatively high MPC into resonators with relatively low MPC. The following is an outline of the essential workings of the program:

STEP 1 Request values from the user for: the mass and charge of the ion, the linac injection energy, and the average cooling power available per resonator

STEP 2 Read in the coefficients describing the forty Q curves from a set of text files.

STEP 3 Set the power dissipation of each resonator to the average-available-power value.

STEP 4 Calculate the accelerating field of each resonator by solving Equation 2 numerically for $\mathcal{E}_n$, using an algorithm based on Newton's method.

STEP 5 Determine the acceleration vector **W** for the entire linac by tracing a particle through one resonator at a time, calculating first the transit-time factor from the velocity and then the acceleration according to Equation 1.

STEP 6 Calculate the marginal power cost M_n for each resonator, and then the average marginal power cost, $M_{av} = \frac{1}{N}\sum_n M_n$.

STEP 7 For each resonator, increase or decrease P_n in proportion to the difference between its marginal power cost and the average, i.e., $P_n := P_n + k(M_{av} - M_n)$ for some appropriate constant k.

STEP 8 Repeat steps 4 through 8 until **P** converges to the optimal power distribution

The convergence of this algorithm rests on the fact that μ_n is always the dominant term in M_n, and that, as noted in Section 3.3 above, μ_n invariably increases with increasing P_n. It follows that raising or lowering P_n always has the effect of raising or lowering M_n. Since the change in each P_n at each iteration is proportional to the difference of M_n from the average of all marginal power costs, the net change in total power P is always 0. More explicitly, the net change in P is given by

$$\Delta P = \sum_n k(M_{av} - M_n) = k(\sum_n M_{av} - \sum_n M_n)$$

$$= k(NM_{av} - \sum_n M_n) = k(\sum_n M_n - \sum_n M_n) = 0.$$

Thus the total power is kept at a fixed value through each iteration, and the marginal power costs are repeatedly moved closer together until they all reach a fixed identical value. The program typically requires about 10 iterations to reach the point at which the achievable acceleration cannot get significantly better.

5 Preliminary results

We have not yet had an opportunity to measure the full set of complete Q curves needed for a definitive test of the optimization procedure. Some preliminary tests have been made with a partially simulated data base in which each incomplete Q curve was assumed to be identical to one of the well-established ones. For typical values of mass, charge and injection energy, the optimum power distribution varies by about ± 2 watts from the initial 6 watts, and the overall acceleration improves by about 2–3%, that is, the equivalent of about one additional resonator. This somewhat smaller than anticipated energy gain improvement could be due in part to coincidental cancellation effects. For example the low-beta QWR resonators generally have somewhat higher Q's than the high-beta SLR's, but they are also shorter (17 cm *versus* 22 cm) and have a somewhat higher specific energy content κ (54 mJ/(MV/m)^2 *versus* 48).

References

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